Midterm Exam Solution

1. The first order condition for consumption is

\[ e^{-bc_i} = \mu p_i \]

and solving for consumption yields

\[ c_i = -\frac{1}{b} \left[ \log(p_i) + \log(\mu) \right]. \]

Substituting into the budget constraint yields

\[ -\frac{1}{b} \log(\mu) = \bar{c} \]

where \( \bar{c} \) is defined in the problem.

2. Firm \( i \) maximizes

\[ p_i \left[ \bar{c} - \frac{1}{b} \log(p_i) \right] \]

The first order condition is

\[ \bar{c} - \frac{1}{b} \log(p_i) - \frac{1}{b} = 0 \]

and so the firm will charge \( p_i = e^{bc^{-1}} \).

3. Using the normalization, profits are given by \( \bar{c} \), and due to free entry profits must go to workers. A firm employs \( \bar{l} \) units of labor, so \( w = \frac{\bar{c}}{\bar{l}} \). The price normalization also implies \( \bar{c} = \frac{1}{b} \), so we get \( w = \frac{1}{b\bar{l}} \). From labor market clearing \( N \frac{\bar{L}}{\bar{l}} = 1 \) we get the number of goods \( N = \frac{\bar{L}}{\bar{l}} \). Income of a worker is given by \( wL = \frac{\bar{L}}{\bar{l}} \) and falls with \( \bar{l} \) as profits have to be distributed among more workers.

4. The utility of a worker is

\[ N \frac{1 - e^{-bc}}{b} = \frac{\bar{l}}{\bar{L}} \frac{1 - \frac{1}{b}}{b} \]

Only the ratio \( \lambda \equiv \frac{\bar{L}}{\bar{l}} \) matters, and higher productivity is associated with higher utility.
5. Managers receive what is left of profits after workers have been paid:

\[ \omega(q) = \frac{1}{b} - w \frac{\bar{l}}{q} \]

6. The left hand side is the supply of workers, the right hand side is the demand for workers. Again only the ratio \( \lambda = \frac{\bar{L}}{\bar{q}} \) matters, and differentiating this yields

\[ \frac{\partial q^*}{\partial \lambda} = \frac{f^{2_{\text{max}}}_q \frac{f(q)}{q} dq}{f(q^*) \left[ 1 + \frac{\lambda}{q^*} \right]} \]

Using the equation implicitly defining \( q^* \) one can also write

\[ \frac{\partial q^*}{\partial \lambda} \frac{\lambda}{q^*} = \frac{F(q^*)}{f(q^*) [q^* + \lambda]} \]

Substituting the assumed distribution function and density yields

\[ \frac{\partial q^*}{\partial \lambda} \frac{\lambda}{q^*} = \frac{q^* - q_{\text{min}}}{q^* + \lambda} < 1, \]

which insures that \( \frac{f}{q} \) is increasing in \( \bar{l} \).

7. The manager with ability \( q^* \) must be indifferent between managing and working, so we must have

\[ wL = \frac{1}{b} - w \frac{\bar{l}}{q^*} \]

and so

\[ w = \frac{1}{b} L + \frac{\bar{l}}{q^*} \]

We can write

\[ wL = \frac{1}{b} \frac{1 + \lambda}{q^*} \]

8. The absolute wage level clearly falls. The number of firms and thus the number of managers is \( N = 1 - F(q^*) \) and falls. If income is \( R \), then utility is

\[ U(R, N) = N \frac{1 - e^{-b \frac{R}{N}}}{b} \]

We already know that the income of a worker \( wL \) falls, which reduces utility. We also know that \( N \) falls. It remains to show that the fall in \( N \) also reduces utility. We have

\[ \frac{\partial U}{\partial N} = \frac{1}{b} q \left( \frac{b R}{N} \right) \]
where
\[ g(x) = 1 - e^{-x} - xe^{-x}. \]
We have \( g(0) = 0 \) and \( g'(x) = 1 + xe^{-x} \), so utility is increasing in \( N \).
We have
\[ \frac{\omega(q)}{wL} = \frac{1}{bwL} \left( 1 + \frac{\lambda}{q^*} - \frac{\lambda}{q} \right). \]
Define
\[ h(\lambda, q^*) = 1 + \lambda \left( \frac{1}{q^*(\lambda)} - \frac{1}{q} \right). \]
We have
\[ \frac{\partial h(\lambda, q^*)}{\partial \lambda} = \frac{1}{q^*(\lambda)} - \frac{1}{q} + \lambda \frac{1}{(q^*)^2} \frac{\partial q^*}{\partial \lambda} \]
Evaluating this at \( q = q^* \) yields
\[ \frac{\partial h(\lambda, q^*)}{\partial \lambda} = -\lambda \frac{1}{(q^*)^2} \frac{\partial q^*}{\partial \lambda} < 0, \]
so inequality between workers and low-quality managers falls. Now
\[ \frac{\partial^2 h(\lambda, q)}{\partial \lambda \partial q} = \frac{1}{q^2} \]
and
\[ \lim_{q \to \infty} \frac{\partial h(\lambda, q)}{\partial \lambda} = \frac{1}{q^*(\lambda)} \left[ 1 - \frac{\partial q^*}{\partial \lambda} \frac{\lambda}{(q^*)} \right] > 0, \]
so inequality between production workers and high-quality managers increases (although \( q_{\text{max}} \) may not be high enough to have an increase in inequality).
Finally we have
\[ \frac{\omega(q')}{\omega(q)} = \frac{h(q', \lambda)}{h(q, \lambda)} \]
and this will be increasing if the elasticity
\[ \frac{\partial h \lambda}{\partial \lambda h} = \frac{\lambda \left( \frac{1}{q^*(\lambda)} - \frac{1}{q} \right) - \lambda^2 \frac{1}{(q^*)^2} \frac{\partial q^*}{\partial x}}{1 + \lambda \left( \frac{1}{q^*(\lambda)} - \frac{1}{q} \right)} \]
is increasing in \( q \). This is the case if
\[ \frac{\lambda}{q^2} \left[ 1 + \lambda \left( \frac{1}{q^*(\lambda)} - \frac{1}{q} \right) \right] - \lambda \frac{1}{q^2} \left[ \lambda \left( \frac{1}{q^*(\lambda)} - \frac{1}{q} \right) - \lambda^2 \frac{1}{(q^*)^2} \frac{\partial q^*}{\partial \lambda} \right] > 0 \]
which is satisfied.