14.462 Lecture Notes
Self Insurance and Risk Taking

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1 Self Insurance

• Exogenous stochastic income stream $y_t$. $y_t$ is i.i.d., with support $[y_{\text{min}}, y_{\text{max}}]$, $y_{\text{max}} > y_{\text{min}} \geq 0$, and c.d.f. $\Psi$.

• Preferences:

$$
E_0 U = E_0 \sum_{t=0}^{\infty} \beta^t U(c_t)
$$

where $U' > 0 > U''$; and, unless otherwise stated, $U'(0) = \infty$, $U''(\infty) = 0$.

• Budget and borrowing constraint:

$$
c_t + a_t = (1 + r)a_{t-1} + y_t
$$

$$
c_t \geq 0
$$

$$
a_t \geq 0
$$

which implies

$$
c_t \leq (1 + r)a_t + y_{t+1}
$$
• Remark: We could relax borrowing constraint to

\[ a_t \geq -\bar{b}_t \]

where \( \bar{b}_t \) is the borrowing limit. Either exogenous to the economy; or endogenous.

E.g.:

\[ \bar{b}_t = \inf_{\{y_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} (1 + r)^{-(j+1)} y_{t+j} = \frac{y_{\text{min}}}{r} \]

• Define cash in hand as

\[ z_t \equiv (1 + r)a_t + y_t \]

It follows that

\[ z_{t+1} = (1 + r)(z_t - c_t) + y_{t+1} \]

and

\[ 0 \leq c_t \leq z_t \]

• We write the Belman equation as:

\[
V(z) = \max_{0 \leq c \leq z} \left[ U(c) + \beta \int V(\tilde{z}) d\Psi(y) \right] \\
\text{s.t. } \tilde{z} = (1 + r)(z - c) + y
\]

We denote by \( C(z) \) the arg max of the above and \( A(z) = z - C(z) \).

• The value function \( V \) is the fixed point of the corresponding operator. Obviously, \( V \) inherits the properties of \( U \). That is, \( V' > 0 > V'' \), \( V'(0) = -\infty \), \( V'(\infty) = 0 \). Also, \( C(z) \) and \( A(z) \) are non-decreasing.

• The FOC:

\[ U'(c_t) \geq \beta (1 + r) \mathbb{E}_t V'(z_{t+1}), \quad \text{if } c_t < z_t \]
The Envelope Condition:

\[ V'(z_t) = U'(c_t) \]

Euler equation:

\[ U'(c_t) \geq \beta(1 + r)\mathbb{E}_t U'(c_{t+1}), \quad \text{if } c_t < z_t \]

Alternatively

\[ V'(z_t) \geq \beta(1 + r)\mathbb{E}_t V'(z_{t+1}), \quad \text{if } \mathbb{E}_t z_{t+1} > (1 + r)z_t + \mathbb{E}_t y_{t+1} \]

1.1 Random Walk and Precautionary Motive

- Consider \( \beta(1 + r) = 1 \), that is, that is, \( r = \rho \equiv \beta^{-1} - 1 \). If there were no uncertainty (and eventually no binding borrowing constraint), then

\[ U'(c_t) = U'(c_{t+1}) \quad \text{or} \quad V'(z_t) = V'(z_{t+1}) \]

implying

\[ c_{t+1} = c_t = c^* \quad \text{and} \quad z_{t+1} = z_t = z^* \]

- Suppose now that there is risk in consumption, but there is no borrowing constraint and \( r = \rho \). Then, the Euler condition implies

\[ \mathbb{E}_t U'(c_{t+1}) = U'(c_t) \quad \text{and} \quad \mathbb{E}_t V'(z_{t+1}) = V'(z_t) \]

If in addition utility is quadratic, implying that \( U' \) and \( V' \) are linear, then

\[ \mathbb{E}_t c_{t+1} = c_t \quad \text{and} \quad \mathbb{E}_t z_{t+1} = z_t \]

That is, consumption and wealth follow a random walk.

- But if \( U'' > 0 \) and \( \text{Var}_t c_{t+1} > 0 \), then \( \mathbb{E}_t U'(c_{t+1}) = U'(c_t) \) implies

\[ \mathbb{E}_t c_{t+1} > c_t \]

The precautionary motive for saving.
1.2 The $U_c$ Supermartingale

- Consider again the general case. Define

$$M_t \equiv \beta^t (1 + r)^t U'(c_t) = \beta^t (1 + r)^t V'(z_t)$$

Then, by the Euler condition,

$$\mathbb{E}_t (M_{t+1} - M_t) \leq 0$$

That is, $M_t$ is a supermartingale. Because $M_t$ is non-negative (actually strictly positive), the supermartingale convergence theorem applies. The latter states that $M_t$ converges almost surely to a non-negative random variable $M_\infty$: $M_t \to a.s. M_\infty$.

- Suppose $\beta(1 + r) > 1$, that is, $r > \rho \equiv \beta^{-1} - 1$. Then, the fact that $M_t$ converges a.s. while $\beta^t (1 + r)^t$ diverges to $+\infty$ implies that $U'(c_t) = V'(z_t)$ must a.s. converge to 0. That is, $c_t$ and $z_t$ diverge a.s. to $+\infty$.

- Suppose next $\beta(1 + r) = 1$, that is, $r = \rho \equiv \beta^{-1} - 1$. We want to argue again that $c_t$ and $z_t$ diverge a.s. to $\infty$. Suppose to the contrary that there is some upper limit $z_{\text{max}} < \infty$ such that $z_{t+1} \leq z_{\text{max}} = (1 + r)A(z_{\text{max}}) + y_{\text{max}}$. At $z_t = z_{\text{max}}$, then

$$V'(z_t) \geq \beta(1 + r)\mathbb{E}_t V'(z_{t+1}) \Rightarrow$$

$$V'(z_{\text{max}}) \geq \mathbb{E}_t V'((1 + r)A(z_{\text{max}}) + y_{t+1})$$

$$> \inf_{y_{t+1}} \{V'((1 + r)A(z_{\text{max}}) + y_{t+1})\} =$$

$$= V'((1 + r)A(z_{\text{max}}) + y_{\text{max}}) = V'(z_{\text{max}}).$$

That is, $V'(z_{\text{max}}) > V'(z_{\text{max}})$, which is a contradiction. The resolution is $\text{Var}_t V'(z_{t+1}) = 0$, which requires either the variance of $y_{t+1}$ to vanish, or otherwise $z_{t+1}$ to diverge a.s. to $+\infty$. 


• Suppose finally $\beta(1 + r) < 1$, that is, $r = \rho \equiv \beta^{-1} - 1$. Then, as long as $\text{Var}_t V'(z_{t+1}) = \text{Var}_t U'(c_{t+1})$ remains finite, then $M_t$ will automatically converge a.s. to zero, and we are fine.

• We conclude that $A(z_0) = \infty$ if $r \geq \rho$, but $A(z_0)$ can be finite if $r < \rho$. With CARA, there is a unique $r < \rho$ for which $A(z_0)$ is finite. With diminishin ARA (such as CRRA), $A(z_0)$ is finite for every $r < \rho$.

• For stochastic $r$, Chamberlain and Wilson (1984/2000) prove that $z$ diverges to infinite as long as $\mathbb{E}r$ exceeds $\rho$.

2 CARA-Normal Example

2.1 Individual behavior

• Suppose $\beta(1 + r) < 1$.

• Suppose $y_t \sim N(\bar{y}, \sigma^2)$.

• Suppose CARA preferences,

$$U(c) = -\frac{1}{\Gamma} \exp(-\Gamma c)$$
$$U'(c) = \exp(-\Gamma c)$$

• Show that there are $a, b, \hat{a}, \hat{b}$ such that

$$V(z) = -\exp(-\hat{a}z - \hat{b})$$
$$C(z) = az + b$$

• Because $c$ is normal and $U'$ is exponential,

$$\mathbb{E}_t U'(c_{t+1}) = U'(\mathbb{E}_tc_{t+1} - \Gamma \text{Var}_t(c_{t+1})/2)$$
• The Euler condition,
\[ U'(c_t) = \beta(1 + r) \mathbb{E}_t U'(c_{t+1}), \]
thus reduces to
\[ \mathbb{E}_t c_{t+1} - c_t = \frac{1}{\Gamma} \ln[\beta(1 + r)] + \frac{\Gamma}{2} \text{Var}_t(c_{t+1}) \]

• Combining with \( C(z) = az + b \) and \( \text{Var}_t(z_{t+1}) = \sigma^2 \), we infer \( \text{Var}_t(c_{t+1}) = a^2 \sigma^2 \) and thus
\[ \mathbb{E}_t c_{t+1} - c_t = \frac{1}{\Gamma} \ln[\beta(1 + r)] + \frac{\Gamma}{2} a^2 \sigma^2 \]

• For a steady state, \( \mathbb{E}_t c_{t+1} - c_t = 0 \), we thus need
\[ \ln[\beta(1 + r)] = -\frac{(\Gamma a \sigma)^2}{2} \]
that is
\[ r = \rho e^{-(\Gamma a \sigma)^2/2} < \rho \]

• Hence, the resolution to the risk-free rate puzzle.

2.2 Moving from CARA to CRRA

• A disturbing property of our CARA specification is that risk aversion is independent of wealth. Indeed, absolute risk aversion is \( \Gamma \), but relative risk aversion is \( \Gamma c_t \). It is more reasonable to assume that relative rather than absolute risk aversion is constant. Therefore, lets us fix \( \Gamma c_t = \gamma \), that is, calibrate \( \Gamma \) as \( \Gamma = \gamma / c_t \), where \( \gamma \) measures relative risk aversion.

• Then, the Euler condition becomes
\[ \frac{\mathbb{E}_t c_{t+1}}{c_t} - 1 = \frac{1}{\gamma} \ln[\beta(1 + r)] + \frac{\gamma}{2} \frac{\text{Var}_t(c_{t+1})}{c_t^2}. \]
Note that $\text{Var}(c_{t+1}) = a^2\sigma^2$, $c_t^2 = (az_t+b)^2 \approx a^2z_t^2$, and $\ln \beta(1+r) \approx r - \rho$ where $\rho \equiv 1/\beta - 1$. Letting $1/\gamma = \theta$ for the elasticity of intertemporal substitution, we conclude

$$\frac{\mathbb{E}_t c_{t+1}}{c_t} = 1 + \theta(r_t - \rho) + \frac{\gamma}{2} \left( \frac{\sigma}{z_t} \right)^2.$$ 

That is, consumption growth (savings) are increasing in the difference between the interest rate and the discount rate and increasing in the income risk relative to the level of wealth.

### 2.3 Towards General Equilibrium

- For consumption and wealth to be stationary, namely $\mathbb{E}_t c_{t+1}/c_t = 1$, we need

$$\theta(r_t - \rho) = -\frac{\gamma}{2} \left( \frac{\sigma}{z_t} \right)^2,$$

which requires $r_t < \rho$. Equivalently,

$$z_t = \sqrt{\frac{\sigma^2/\gamma}{2\theta(\rho - r_t)}} \equiv Z(r_t).$$

- $Z(r)$ corresponds to the aggregate supply of savings: It says what is the stationary level of wealth for any given interest rate. Note that $Z(r) \in (0, \infty)$ and $Z'(r) > 0$ for all $r \in [0, \rho)$, with $Z(0) < \infty$ and $Z(r) \to \infty$ as $r \to \rho$.

- On the other hand, the optimal level of investment is pinned down by the equality of the MPK with the interest rate:

$$r_t = f'(K_t).$$

Equivalently,

$$k_t = (f')^{-1}(r_t) \equiv K(r_t).$$

- $K(r)$ corresponds to the aggregate demand for capital. Note that $K(r) \in (0, \infty)$ and $K'(r) < 0$ for all $r \in (0, \rho]$, with $K(r) \to \infty$ as $r \to 0$ and $K(\rho) < \infty$. 
• A steady state corresponds to an intersection of the curves $Z(r)$ and $K(r)$. That is, the steady-state interest rate and capital stock are given by $r^*$ and $k^*$ such that $Z(r^*) = K(r^*) = k^*$.

• By the properties of $Z$ and $K$, the steady state exists and is unique.

• Moreover, for any $\sigma > 0$, the steady state is $r^* < \rho$ and $k^* > K(\rho)$. That is, the interest rate is lower and the capital stock is higher under incomplete markets than under complete markets.

• Finally, an increase in $\sigma$ (labor income risk) increases the supply of savings $Z(r)$ without affecting the demand for investment $K(r)$. Therefore, $r^*$ is decreasing in $\sigma$, and $k^*$ is increasing in $\sigma$.

• The above analysis is a heuristic representation of the more formal and exact analysis in Aiyagari (1994).