Problem Set 6

14.462 Topics in Macro

Spring 2007

Problem 1
Consider the Holmstrom and Tirole model we have seen in class (see slides for notation). We embed it in the following general equilibrium environment. The production function is $AF(K, L)$, it is strictly concave and displays constant returns to scale. Consumers/workers receive the constant endowment ω in period 0 and wages w in period 1. Labor supply is 1. Let K^* be such that

$$p^h A^H F_1\left(K^*, 1/p^h\right) = 1 + e^h,$$

and $\hat{K}_1$ such that

$$p^l A^H F_1\left(\hat{K}_1, 1/p^l\right) = 1 + e^l.$$

Make the following assumptions: (A) $\hat{K}_1 < K^*$, (B) F satisfies Inada conditions.
Let $\tilde{K}$ be the value such that

$$\tilde{K} = p^h \left[A^H F_1\left(\tilde{K}, 1/p^h\right) - \frac{\Delta e}{\Delta p}\right] \tilde{K} + N.$$

(i) Show that for each N there is a unique $\tilde{K} > N$ that solves this equation. Show that $\tilde{K}$ is increasing in N.

(ii) Suppose that $\tilde{K} \leq \hat{K}_1$. Show that there is an equilibrium where $e = e^l$, $K = \hat{K}_1$ and the entrepreneur's financial constraint is slack.

(iii) Argue that, in general, there can be no equilibrium with $K < \hat{K}_1$.

(iv) Suppose that $\tilde{K} > K^*$. Show that there is an equilibrium where $e = e^h$ and $K = K^*$.

(v) Suppose that $\tilde{K} \in [\hat{K}_1, K^*)$. Show that there is an equilibrium where $e = e^h$, $K = \tilde{K}$.

Note: for (ii), (iv) and (v) check that the equilibrium action is indeed optimal.

Suppose the consumers make a voluntary transfer τ to the entrepreneurs, so that they begin life with $\omega - \tau$ and the entrepreneurs with $N + \tau$. Suppose the production function is $F(K, L) = K^\alpha L^{1-\alpha}$.

(vi) Plot the utility possibility frontier for your choice of parameters. (Numerical results are ok.)

(vii) Find parameters such that $\tau > 0$ leads to a Pareto improvement. (Same here).

Problem 2 (Tirole 2006 3.12)

One of the key developments in the theory of market finance has been to find methods to price claims held by investors. Market finance emphasizes state-contingent pricing, the fact that 1 unit of income does not have a uniform value across states of nature. We have assumed that investors are risk neutral, and so it does not matter how the pledgeable income is spread across states of nature. This assumption is made only for the sake of computational simplicity, and can easily be relaxed. Consider a two-date model of market finance with a representative consumer/investor. This consumer has utility of consumption $u(c_0)$ at date 0, the date at which he lends to the firm, and utility of consumption $u(c(\omega))$ at date 1, date at which he receives the return from investment. There is macroeconomic uncertainty in that the representative consumer's date-1 consumption depends on the state of nature ω. The state of nature described both what happens in this particular firm and in the rest of the economy (even though aggregate consumption is independent of the outcome in this particular firm to the extent that the firm is atomistic, which we will assume).

Suppose that the entrepreneur works. Let S denote the even that "the project succeeds" and F the even that "the project fails". Let

$$\begin{aligned} q_S &= \mathbf{E}\left[\frac{u'(c(\omega))}{u'(c_0)} | \omega \in S\right] \text{ and} \\ q_F &= \mathbf{E}\left[\frac{u'(c(\omega))}{u'(c_0)} | \omega \notin S\right]. \end{aligned}$$

The firm's activity is said to covary positively with the economy (be "procyclical") if $q_S < q_F$, and negatively (be "countercyclical") if $q_F < q_S$.

Suppose that

$$p_H q_S + (1 - p_H) q_F = 1.$$

(i) Interpret this assumption.

(ii) Consider a fixed-investment version of the model from class. A project requires investment I and the entrepreneur has initial net work of $A < I$. If undertaken, the project either succeeds and yields verifiable income $R > 0$ or it fails and yields no income. If the entrepreneur behaves, the probability of success is p_H and if he shirks, he receives private benefit $B > 0$ and the probability of success is $p_L < p_H$. Derive the necessary and sufficient condition for the project to receive financing.

(iii) What is the optimal contract between the investors and the entrepreneurs? Does it involve maximum punishment ($R_b = 0$) in the case of failure? How would your answer change if the entrepreneur were risk averse? (For simplicity, assume that her only claim is in the firm. She does not hold any of the market portfolio).

Problem 3 (Tirole 2006 4.10)

An entrepreneur has two variable-investment projects $i = \{1, 2\}$. For investment level $I^i \in [0, \infty)$, project i yields RI^i in the case of success and 0 in the case of failure. The probability of success is p_H if the entrepreneur behaves (and thereby gets no private benefit) and $p_L = p_H - \Delta p$ if she misbehaves (and obtains private benefit BI^i). Universal risk neutrality prevails and the entrepreneur is protected by limited liability. The two projects are independent (not correlated). The entrepreneur starts with total wealth A. Assume

$$\begin{aligned} \rho_1 &= p_H R > 0 > \rho_0 = p_H \left(R - \frac{B}{\Delta p} \right) \text{ and} \\ \rho_0' &= p_H \left(R - \frac{p_H}{p_H + p_L} \frac{B}{\Delta p} \right) < 1. \end{aligned}$$

(i) First, consider project finance (each project is financed on a stand-alone basis). Compute the borrower's utility. Is there any benefit from having access to two projects rather than one?

(ii) Compute the borrower's utility under cross-pledging.