Lecture Note on Dynamic Insurance

November 2012

• Atkeson-Lucas:
  – basic model of dynamic insurance
  – surprising implication: immiseration

• preferences
  \[
  \mathbb{E}_{-1} \sum_{t=0}^{\infty} \beta^t \theta_t U(c_t) = \sum_{t, \theta^t} \beta^t \theta_t U(c(\theta^t)) \Pr(\theta^t)
  \]
  \[
  U(c) = \frac{c^{1-\sigma}}{1-\sigma}
  \]

• \( \theta \in \Theta \) finite

• \( \theta_t \) is i.i.d. with density \( p(\theta) \) with \( \sum_\theta p(\theta) = 1; \Pr(\theta^t) = p(\theta_0)p(\theta_1) \cdots p(\theta_t) \)

• resource constraints
  \[
  \sum_{t, \theta^t} c(\theta^t) \Pr(\theta^t) \leq e \quad t = 0, 1, \ldots
  \]

• First best:
  – FOC
    \[
    \theta_t U'(c(\theta^t)) = \lambda_t
    \]
    and resource constraint with equality
    \[
    \implies c(\theta^t) = g(\theta_t)
    \]
    where
    \[
    \theta U'(g(\theta)) = \bar{\lambda}
    \]
- history independent
- consumption rises with $\theta_t$
- not incentive compatible: everyone would report to have the highest shock

**Incentive compatibility:**

- direct mechanism:
  * reports $r_t \in \Theta$, history of reports $r^t \in \Theta^{t+1}$
  * allocation $c(r^t)$
  * strategy $r_t = \sigma_t(\theta^t)$; induces history $r^t = \sigma^t(\theta^t)$
- truth-telling (IC)

\[
\sum_{t,\theta^t} \beta^t \theta_t U(c(\theta^t)) \Pr(\theta^t) \geq \sum_{t,\theta^t} \beta^t \theta_t U(c(\sigma^t(\theta^t))) \Pr(\theta^t) \quad \forall \sigma
\]

**Second best planning problem:**

\[
\max \sum_{t,\theta^t} \beta^t \theta_t U(c(\theta^t)) \Pr(\theta^t)
\]

subject to IC and RC.

**Approach:**

- study dual
- relax dual
- recursive formulation

**Dual:**

\[
\min e
\]

s.t. IC and

\[
\sum_{t,\theta^t} \beta^t \theta_t U(c(\theta^t)) \Pr(\theta^t) = v_0
\]

\[
\sum_{\theta^t} c(\theta^t) \Pr(\theta^t) \leq e
\]
• relaxed dual: replace RC with

\[
\sum_{t=0}^{\infty} q^t \sum_{\theta^t} c(\theta^t) \Pr(\theta^t) \leq \sum_{t=0}^{\infty} q^t e
\]

for some \( q \in (0, 1) \)

• Full statement

\[
K(v) = \min \sum_{t=0}^{\infty} q^t \sum_{\theta^t} c(\theta^t) \Pr(\theta^t)
\]

s.t. PK

\[
\sum_{t,\theta^t} \beta^t \theta^t U(c(\theta^t)) \Pr(\theta^t) = v_0
\]

and IC

\[
\sum_{t,\theta^t} \beta^t \theta^t U(c(\theta^t)) \Pr(\theta^t) \geq \sum_{t,\theta^t} \beta^t \theta^t U(c(\sigma^t(\theta^t))) \Pr(\theta^t) \quad \forall \sigma
\]

• utility assignments:

  – chose \( u(\theta^t) \) instead of \( c(\theta^t) \)
  
  – let \( C = U^{-1} \)

• Same problem:

\[
K(v) = \min \sum_{t=0}^{\infty} q^t \sum_{\theta^t} C(u(\theta^t)) \Pr(\theta^t)
\]

\[
\sum_{t,\theta^t} \beta^t \theta^t u(\theta^t) \Pr(\theta^t) = v_0
\]

\[
\sum_{t,\theta^t} \beta^t \theta^t u(\theta^t) \Pr(\theta^t) \geq \sum_{t,\theta^t} \beta^t \theta^t u(\sigma^t(\theta^t))) \Pr(\theta^t) \quad \forall \sigma
\]

• it follows that \( K(v) \) is convex and indeed homogeneous:

\[
K(v) = A[(1 - \sigma)v]^{\frac{1}{1 - \sigma}}
\]

• recursive version:

  – continuation utility

\[
v(\theta^{t-1}) = E_{t-1} \sum_{\tau=0}^{\infty} \beta^\tau \theta_{t+\tau} u(c_{t+\tau})
\]
then
\[ v(\theta^{t-1}) = \sum_{\theta_t \in \Theta} [\theta u(c(\theta^{t-1}, \theta_t)) + \beta v(\theta^{t-1}, \theta_t)] p(\theta) \]

- recursive version (drop history notation)
\[ v = \sum_{\theta \in \Theta} [\theta u(c(\theta)) + \beta w(\theta)] p(\theta) \]

- temporary incentive constraint
\[ \theta u(c(\theta)) + \beta w(\theta) \geq \theta u(c(\theta')) + \beta w(\theta') \quad \forall \theta, \theta' \]

- Planning problem
\[ K(v) = \min_{t=0}^{\infty} \sum_{t=0}^{\infty} q^t \sum_{\theta} [C(\theta(\theta)) + \beta K(w(\theta))] \Pr(\theta^t) \]
\[ v = \sum_{\theta \in \Theta} [\theta u(\theta) + \beta w(\theta)] p(\theta) \]
\[ \theta u(\theta) + \beta w(\theta) \geq \theta u(\theta') + \beta w(\theta') \quad \forall \theta, \theta' \]

- policy functions
\[ u(\theta) = \bar{g}^u(\theta, v) \]
\[ w(\theta) = \bar{g}^w(\theta, v) \]

- homogeneity implies
\[ u(\theta) = \bar{g}^u(\theta, v) = \bar{g}^u(\theta)v \]
\[ w(\theta) = \bar{g}^w(\theta, v) = \bar{g}^w(\theta)v \]

- geometric random walk!

- implication: inequality is ever expanding

- consumption
\[ C(u(\theta)) = \bar{g}^u(\theta) \frac{1}{1-\sigma} ((1 - \sigma)v)^{\frac{1}{1-\sigma}} \]
• average consumption:

\[ \left( \sum_{\theta} \tilde{g}^{u}(\theta) \frac{1}{1-\sigma} p(\theta) \right) ((1-\sigma)v)^{\frac{1}{1-\sigma}} \]

• Q: relaxed problem solves original?

• A: Yes. find \( q \) such that

\[ \mathbb{E}_{t-1} c_t = \mathbb{E}_{t-1} c_{t+1} \]

\[ x_{t-1} \equiv ((1-\sigma)v_{t-1})^{\frac{1}{1-\sigma}} = \mathbb{E}_{t-1} (((1-\sigma)v_t)^{\frac{1}{1-\sigma}} = \mathbb{E}_{t-1} x_t \]

this requires

\[ ((1-\sigma)v)^{\frac{1}{1-\sigma}} = ((1-\sigma)\tilde{g}^{w}(\theta)v)^{\frac{1}{1-\sigma}} \]

\[ 1 = \tilde{g}^{w}(\theta)^{\frac{1}{1-\sigma}} \]

• immiseration:

\[ x_t = \epsilon_t x_{t-1} \]

\[ \mathbb{E}_{t-1} \epsilon = 1 \]

• implication: (Martingale convergence theorem)

\[ x_t \to 0 \quad \text{a.s.} \]