Consider an economy (Home) with 2 goods, Cloth ($C$) and Food ($F$). All consumers have same Cobb-Douglas utility function:

$$U(D_C, D_F) = (D_C)^{1-\alpha} (D_F)^\alpha$$  \hspace{1cm} (1)

with $\alpha \in (0,1)$. The production technologies are:

$$F_F(K_F, L_F) = (K_F)^{\beta_F} (L_F)^{1-\beta_F}$$  \hspace{1cm} (2)

$$F_C(K_C, L_C) = (K_C)^{\beta_C} (L_C)^{1-\beta_C}$$  \hspace{1cm} (3)

with $\beta_F, \beta_C \in (0,1)$ and $\beta_F > \beta_C$.

1. The objective of this first exercise is to demonstrate the Stolper-Samuelson Theorem when Equations (2)-(3) hold.

(a) Let $a_{K_i}$ and $a_{L_i}$ denote the amount of capital and labor used in one unit of good $i = C, F$. Show that:

$$\frac{a_{K_i}}{a_{L_i}} = \frac{\beta_i}{(1-\beta_i)} \frac{w}{r}$$  \hspace{1cm} (4)

(b) Using Equation (4), show that:

$$a_{L_i} = \left[ \frac{\beta_i}{(1-\beta_i)} \cdot \frac{w}{r} \right]^{-\beta_i}$$

$$a_{K_i} = \left[ \frac{\beta_i}{(1-\beta_i)} \cdot \frac{w}{r} \right]^{-\beta_i+1}$$

(c) Let $p_i$ denote the price of good $i = C, F$. Show that:

$$p_i = \beta_i^{\beta_i} (1-\beta_i)^{\beta_i-1} r^\beta_i w^{1-\beta_i}$$  \hspace{1cm} (5)

(d) Show that an decrease in $p = p_C/p_F$ increases the real return of capital and decreases the real return of labor [Stolper-Samuelson Theorem].

2. The objective of this second exercise is to demonstrate the Rybczynski Theorem when Equations (2)-(3) hold.
(a) Let \( Q_C \) and \( Q_F \) denote the output of good \( C \) and \( F \). Show that

\[
Q_C = \frac{L - \frac{\alpha_F}{\alpha_K} K}{\frac{\alpha_L}{\alpha_K} \left( \frac{\alpha_L}{\alpha_K} - \frac{\alpha_F}{\alpha_K} \right)}
\]  

(6)

\[
Q_F = \frac{\frac{\alpha_L}{\alpha_K} K - L}{\frac{\alpha_F}{\alpha_K} \left( \frac{\alpha_L}{\alpha_K} - \frac{\alpha_F}{\alpha_K} \right)}
\]  

(7)

(b) Using Equation (4), show that an increase in \( K \) raises \( Q_F \) and lowers \( Q_C \).

3. The objective of this last exercise is to demonstrate the Heckscher-Ohlin Theorem when Equations (1)-(3).

(a) Using Equation (1), show that

\[
\frac{p_C Q_C}{p_F Q_F} = \frac{1 - \alpha}{\alpha}
\]

(b) Using Equation (5), show that

\[
\frac{\beta_F (1 - \beta_F)^{1-\beta_F}}{\beta_C (1 - \beta_C)^{1-\beta_C}} \left( \frac{w}{r} \right)^{\beta_F - \beta_C} \frac{Q_C}{Q_F} = \frac{1 - \alpha}{\alpha}
\]

(8)

(c) Using Equations (6), (7), and (8), show that labor-abundant countries tend to produce disproportionate amount of Cloth [Heckscher-Ohlin Theorem].