Optimization Problems,

John Guttag

MIT Department of Electrical Engineering and Computer Science
Relevant Reading for Today’s Lecture

- Chapter 13
The Pros and Cons of Greedy

- Easy to implement
- Computationally efficient

- But does not always yield the best solution
  - Don’t even know how good the approximation is

Question 1
1. Enumerate all possible combinations of items.

2. Remove all of the combinations whose total units exceed the allowed weight.

3. From the remaining combinations choose any one whose value is the largest.
Search Tree Implementation

- The tree is built top down starting with the root
- The first element is selected from the still to be considered items
  - If there is room for that item in the knapsack, a node is constructed that reflects the consequence of choosing to take that item. By convention, we draw that as the left child
  - We also explore the consequences of not taking that item. This is the right child
- The process is then applied *recursively* to non-leaf children
- Finally, chose a node with the highest value that meets constraints
A Search Tree Enumerates Possibilities

Left-first, depth-first enumeration

Val = 170
Cal = 766

Val = 120
Cal = 766

Val = 140
Cal = 508

Val = 90
Cal = 145

Val = 80
Cal = 612

Val = 30
Cal = 258

Val = 50
Cal = 354

Val = 0
Cal = 0
Computational Complexity

- Time based on number of nodes generated
- Number of levels is number of items to choose from
- Number of nodes at level $i$ is $2^i$
- So, if there are $n$ items the number of nodes is
  - $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
  - i.e., $O(2^n + 1)$

- An obvious optimization: don’t explore parts of tree that violate constraint (e.g., too many calories)
  - Doesn’t change complexity

- Does this mean that brute force is never useful?
  - Let’s give it a try
def maxVal(toConsider, avail):
    """Assumes toConsider a list of items,
    avail a weight
    Returns a tuple of the total value of a
    solution to 0/1 knapsack problem and
    the items of that solution"""

toConsider. Those items that nodes higher up in the tree
(corresponding to earlier calls in the recursive call stack)
have not yet considered

avail. The amount of space still available
Body of maxVal (without comments)

if toConsider == [] or avail == 0:
    result = (0, ())
elif toConsider[0].getUnits() > avail:
    result = maxVal(toConsider[1:], avail)
else:
    nextItem = toConsider[0]
    withVal, withToTake = maxVal(toConsider[1:],
        avail - nextItem.getUnits())
    withVal += nextItem.getValue()
    withoutVal, withoutToTake = maxVal(toConsider[1:], avail)
    if withVal > withoutVal:
        result = (withVal, withToTake + (nextItem,))
    else:
        result = (withoutVal, withoutToTake)
return result

Does not actually build search tree
Local variable result records best solution found so far
Try on Example from Lecture 1

- With calorie budget of 750 calories, chose an optimal set of foods from the menu

<table>
<thead>
<tr>
<th>Food</th>
<th>wine</th>
<th>beer</th>
<th>pizza</th>
<th>burger</th>
<th>fries</th>
<th>coke</th>
<th>apple</th>
<th>donut</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>89</td>
<td>90</td>
<td>30</td>
<td>50</td>
<td>90</td>
<td>79</td>
<td>90</td>
<td>10</td>
</tr>
<tr>
<td>calories</td>
<td>123</td>
<td>154</td>
<td>258</td>
<td>354</td>
<td>365</td>
<td>150</td>
<td>95</td>
<td>195</td>
</tr>
</tbody>
</table>
Search Tree Worked Great

- Gave us a better answer
- Finished quickly
- But $2^8$ is not a large number
  - We should look at what happens when we have a more extensive menu to choose from
import random

def buildLargeMenu(numItems, maxVal, maxCost):
    items = []
    for i in range(numItems):
        items.append(Food(str(i),
                          random.randint(1, maxVal),
                          random.randint(1, maxCost)))
    return items

for numItems in (5,10,15,20,25,30,35,40,45,50,55,60):
    items = buildLargeMenu(numItems, 90, 250)
    testMaxVal(items, 750, False)
Is It Hopeless?

- In theory, yes
- In practice, no!
- Dynamic programming to the rescue
Sometimes a name is just a name

“The 1950s were not good years for mathematical research... I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics... What title, what name, could I choose? ... It's impossible to use the word dynamic in a pejorative sense. Try thinking of some combination that will possibly give it a pejorative meaning. It's impossible. Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities.

-- Richard Bellman
Recursive Implementation of Fibonacci

def fib(n):
    if n == 0 or n == 1:
        return 1
    else:
        return fib(n - 1) + fib(n - 2)

fib(120) = 8,670,007,398,507,948,658,051,921
Call Tree for Recursive Fibonacci

fib(6)
Clearly a Bad Idea to Repeat Work

- Trade a time for space
- Create a table to record what we’ve done
  - Before computing \( \text{fib}(x) \), check if value of \( \text{fib}(x) \) already stored in the table
  - If so, look it up
  - If not, compute it and then add it to table
- Called \text{memoization}
Using a Memo to Compute Fibonacci

def fastFib(n, memo = {}):
    """Assumes n is an int >= 0, memo used only by recursive calls
    Returns Fibonacci of n"
    if n == 0 or n == 1:
        return 1
    try:
        return memo[n]
    except KeyError:
        result = fastFib(n-1, memo) +
        fastFib(n-2, memo)
        memo[n] = result
    return result
When Does It Work?

- **Optimal substructure**: a globally optimal solution can be found by combining optimal solutions to local subproblems
  - For $x > 1$, $\text{fib}(x) = \text{fib}(x - 1) + \text{fib}(x - 2)$

- **Overlapping subproblems**: finding an optimal solution involves solving the same problem multiple times
  - Compute $\text{fib}(x)$ or many times
What About 0/1 Knapsack Problem?

- Do these conditions hold?

Questions 2 and 3
Search Tree

Optimal substructure?
Overlapping subproblems?

Take

Don’tTake

Val = 170
Cal = 766

Val = 120
Cal = 766

Val = 140
Cal = 508

Val = 90
Cal = 145

Val = 80
Cal = 612

Val = 30
Cal = 258

Val = 50
Cal = 354

Val = 0
Cal = 0
A Different Menu
**Need Not Have Copies of Items**

<table>
<thead>
<tr>
<th>Item</th>
<th>Value</th>
<th>Calories</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>b</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>c</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>d</td>
<td>9</td>
<td>5</td>
</tr>
</tbody>
</table>
Search Tree

- Each node = <taken, left, value, remaining calories>
What Problem is Solved at Each Node?

- Given remaining weight, maximize value by choosing among remaining items
- Set of previously chosen items, or even value of that set, doesn’t matter!
Overlapping Subproblems
Modify maxVal to Use a Memo

- Add memo as a third argument
  - `def fastMaxVal(toConsider, avail, memo = {}):`

- Key of memo is a tuple
  - `(items left to be considered, available weight)`
  - `Items left to be considered represented by len(toConsider)`

- First thing body of function does is check whether the optimal choice of items given the available weight is already in the memo

- Last thing body of function does is update the memo
<table>
<thead>
<tr>
<th>len(items)</th>
<th>$2^{**}$len(items)</th>
<th>Number of calls</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>25</td>
</tr>
<tr>
<td>8</td>
<td>256</td>
<td>427</td>
</tr>
<tr>
<td>16</td>
<td>65,536</td>
<td>5,191</td>
</tr>
<tr>
<td>32</td>
<td>4,294,967,296</td>
<td>22,701</td>
</tr>
<tr>
<td>64</td>
<td>18,446,744,073,709,551,616</td>
<td>42,569</td>
</tr>
<tr>
<td>128</td>
<td>Big</td>
<td>83,319</td>
</tr>
<tr>
<td>256</td>
<td>Really Big</td>
<td>176,614</td>
</tr>
<tr>
<td>512</td>
<td>Ridiculously big</td>
<td>351,230</td>
</tr>
<tr>
<td>1024</td>
<td>Absolutely huge</td>
<td>703,802</td>
</tr>
</tbody>
</table>
How Can This Be?

- Problem is exponential
- Have we overturned the laws of the universe?
- Is dynamic programming a miracle?
- No, but computational complexity can be subtle
- Running time of fastMaxVal is governed by number of distinct pairs, \(<\text{toConsider}, \text{avail}>\)
  - Number of possible values of toConsider bounded by \(\text{len(items)}\)
  - Possible values of avail a bit harder to characterize
    - Bounded by number of distinct sums of weights
    - Covered in more detail in assigned reading
Many problems of practical importance can be formulated as optimization problems.

Greedy algorithms often provide adequate (though not necessarily optimal) solutions.

Finding an optimal solution is usually exponentially hard.

But dynamic programming often yields good performance for a subclass of optimization problems—those with optimal substructure and overlapping subproblems:
  - Solution always correct
  - Fast under the right circumstances
The “Roll-over” Optimization Problem

Score = ((60 − (a+b+c+d+e))*F + a*ps1 + b*ps2 + c*ps3 + d*ps4 + e*ps5

Objective:
Given values for F, ps1, ps2, ps3, ps4, ps5
Find values for a, b, c, d, e that maximize score

Constraints:
a, b, c, d, e are each 10 or 0
a + b + c + d + e ≥ 20