The Impedance Model
Sinusoidal Steady State (SSS) Reading 13.1, 13.2

Focus on steady state, only care about $v_P$ as $v_H$ dies away.

Focus on sinusoids.

Sinusoidal Steady State (SSS) Reading 13.1, 13.2

Reading: Section 13.3 from course notes.
Review

$V_i \cos \omega t$

1. usual circuit model
2. set up DE
3. nightmare trig.
4. total

The Sneaky Path

Set up DE usual circuit model

Complex algebra

Take real part

$V_p$

contains all the information we need:

$|V_p| \quad \text{Amplitude of output cosine}$

$\angle V_p \quad \text{phase}$
Review

\[ v_O = |V_p| \cos(\omega t + \angle V_p) \]

\[ V_p \over V_i = {1 \over 1 + j\omega RC} = H(j\omega) \]

\[ \left| \frac{V_p}{V_i} \right| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \]

Bode plot

Break frequency

\[ \omega = \frac{1}{RC} \]

\[ \angle \left( \frac{V_p}{V_i} \right) \]

\[ \tan^{-1}\left( \frac{-\omega RC}{1} \right) = \frac{\pi}{4} \]

The Frequency View

Cite as: Anant Agarwal and Jeffrey Lang, course materials for 6.002 Circuits and Electronics, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].
Is there an even simpler way to get $V_p$?

$$V_p = \frac{V_i}{1 + j\omega RC}$$

Divide numerator and denominator by $j\omega C$.

$$V_p = V_i \frac{1}{\frac{1}{j\omega C} + R}$$

Hmmm... looks like a voltage divider relationship.

$$V_p = V_i \frac{Z_C}{Z_C + R}$$

Let's explore further...
The Impedance Model

Is there an even simpler way to get $V_p$?

Consider:

- **Resistor**
  \[ i_R = I_r e^{j \omega t} \quad v_R = R i_R \]
  \[ V_R = R I_r \]

- **Capacitor**
  \[ i_C = I_C e^{j \omega t} \quad v_C = C \frac{d v_C}{dt} \]
  \[ V_C = \left( \frac{1}{j \omega C} \right) I_C \]

- **Inductor**
  \[ i_L = I_l e^{j \omega t} \quad v_L = L \frac{d i_L}{dt} \]
  \[ V_L = (j \omega L) I_l \]
In other words,

- **Capacitor**
  \[ + \quad V_c \quad Z_C \quad - \]
  \[ I_c \quad V_c = Z_C I_c \]
  \[ Z_C = \frac{1}{j\omega C} \]

- **Inductor**
  \[ + \quad V_l \quad Z_L \quad - \]
  \[ I_l \quad V_l = Z_l I_l \]
  \[ Z_l = j\omega L \]

- **Resistor**
  \[ + \quad V_r \quad Z_R \quad - \]
  \[ I_r \quad V_r = Z_r I_r \]
  \[ Z_r = R \]

For a drive of the form \( V_c e^{j\omega t} \), complex amplitude \( V_c \) is related to the complex amplitude \( I_c \) algebraically, by a generalization of Ohm's Law.
Back to RC example...

Impedance model:

\[ Z_R = R \]

\[ Z_C = \frac{1}{j\omega C} \]

\[ V_c = \frac{j\omega C}{l + R} \]

\[ V_i = \frac{Z_C}{Z_C + Z_R} V_i \]

\[ V_c = \frac{1}{l + j\omega RC} V_i \]

All our old friends apply!
KVL, KCL, superposition...

Done!
Another example, recall series RLC:

Remember, we want only the steady-state response to sinusoid

\[ V_r = \frac{V_i Z_R}{Z_L + Z_C + Z_R} \]

\[ V_r = \frac{V_i R}{j \omega L + \frac{1}{j \omega C} + R} \]

\[ V_r = \frac{V_i j \omega CR}{-\omega^2 LC + 1 + j \omega CR} \]

We will study this and other functions in more detail in the next lecture.
The Big Picture...

\[ V_i \cos \omega t \]

usual circuit model \rightarrow set up DE \rightarrow nightmare trig.

\[ |V_p| \cos [\omega t + \angle V_p] \]
The Big Picture...

\[ V_i \cos \omega t \]

usual circuit model

\[ |V_p| \cos[\omega t + \angle V_p] \]

set up DE

nightmare trig.

\[ V_i e^{j\omega t} \]

drive

complex algebra

take real part
The Big Picture...

$V_i \cos \omega t$  $|V_p| \cos[\omega t + \angle V_p]$

No D.E.s, no trig!
Let's study this transfer function

\[
\frac{V_r}{V_i} = \frac{j \omega RC}{1 + j \omega RC - \omega^2 LC}
\]

\[
= \frac{j \omega RC}{(1 - \omega^2 LC) + j \omega RC} \cdot \frac{(1 - \omega^2 LC) - j \omega RC}{(1 - \omega^2 LC) - j \omega RC}
\]

\[
\left| \frac{V_r}{V_i} \right| = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}
\]

Observe

Low \( \omega \): \( \approx \omega RC \)

High \( \omega \): \( \approx \frac{R}{\omega L} \)

\( \omega \sqrt{LC} = 1 \): \( \approx 1 \)
Graphically

\[
\left| \frac{V_r}{V_i} \right| = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}
\]

Low \( \omega \): \( \approx \omega RC \)
High \( \omega \): \( \approx \frac{R}{\omega L} \)
\( \omega \sqrt{LC} = 1 \): \( \approx 1 \)

Remember this trick to sketch the form of transfer functions quickly.

More next week...