6.003 Homework #7 Solutions

Problems

1. Second-order systems

The impulse response of a second-order CT system has the form

\[ h(t) = e^{-\sigma t} \cos(\omega_d t + \phi) u(t) \]

where the parameters \( \sigma, \omega_d, \) and \( \phi \) are related to the parameters of the characteristic polynomial for the system: \( s^2 + Bs + C \).

a. Determine expressions for \( \sigma \) and \( \omega_d \) (not \( \phi \)) in terms of \( B \) and \( C \).

Express the impulse response in terms of complex exponentials:

\[ h(t) = \frac{1}{2} e^{-\sigma t} \left( e^{j\omega_d t + j\phi} + e^{-j\omega_d t - j\phi} \right) u(t) = \frac{1}{2} e^{j\phi} e^{(-\sigma + j\omega_d)t} u(t) + \frac{1}{2} e^{-j\phi} e^{(-\sigma - j\omega_d)t} u(t) \]

The impulse response is a weighted sum of modes of the form \( e^{s_0 t} \) and \( e^{s_1 t} \) where \( s_0 \) and \( s_1 \) are the poles. Thus the poles of the system are at \( s = -\sigma \pm j\omega_d \). The characteristic polynomial has the form \( s^2 + Bs + C \). Thus \( B = 2\sigma \) and \( C = \sigma^2 + \omega_d^2 \).

Solving, we find that

\[ \sigma = \frac{B}{2} \]
\[ \omega_d = \sqrt{C - \frac{1}{4}B^2}. \]

b. Determine

– the time required for the envelope \( e^{-\sigma t} \) of \( h(t) \) to diminish by a factor of \( e \),
– the period of the oscillations in \( h(t) \), and
– the number of periods of oscillation before \( h(t) \) diminishes by a factor of \( e \).

Express your results as functions of \( B \) and \( C \) only.

The time to decay by a factor of \( e \) is

\[ \frac{1}{\sigma} = \frac{2}{B}. \]

The period is

\[ \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{C - \frac{1}{4}B^2}}. \]

The number of periods before diminishing a factor of \( e \) is

\[ \frac{2\pi}{\sqrt{C - \frac{1}{4}B^2}} = \frac{\sqrt{C - \frac{1}{4}B^2}}{\pi B}. \]

Notice that this last answer is equivalent to \( Q/\pi \) where \( Q = \frac{\omega_d}{2\sigma} \).
c. Estimate the parameters in part b for a CT system with the following poles:

\[
\begin{align*}
\times & -100 \text{ s-plane} \\
\end{align*}
\]

From the plot \( \sigma = 10 \) and \( \omega_d = 100 \).

The time to decay by a factor of \( e \) is 0.1.

The period is \( \frac{2\pi}{\omega_d} = \frac{2\pi}{100} = 0.0628 \).

The number of cycles before decaying by \( e \) is \( \frac{10}{2\pi} \approx 1.6 \).

The unit-sample response of a second-order DT system has the form

\[
h[n] = r_0^n \cos(\Omega_0 n + \Phi) u[n]
\]

where the parameters \( r_0 \), \( \Omega_0 \), and \( \Phi \) are related to the parameters of the characteristic polynomial for the system: \( z^2 + Dz + E \).

d. Determine expressions for \( r_0 \) and \( \Omega_0 \) (not \( \Phi \)) in terms of \( D \) and \( E \).

Express the unit-sample response in terms of complex exponentials:

\[
h[n] = r_0^n \left( \frac{1}{2} e^{j\Omega_0 n + j\Phi} + \frac{1}{2} e^{-j\Omega_0 n - j\Phi} \right) u[n] = \frac{1}{2} e^{j\Phi} r_0^n e^{j\Omega_0 n} u[n] + \frac{1}{2} e^{-j\Phi} r_0^n e^{-j\Omega_0 n} u[n]
\]

The poles have the form \( z = r_0 e^{j\Omega_0} \) and \( z = r_0 e^{-j\Omega_0} \). The characteristic equation is \( z^2 + Dz + E = (z - r_0 e^{j\Omega_0})(z - r_0 e^{-j\Omega_0}) = z^2 - 2r_0 \cos \Omega_0 + r_0^2 \). Thus \( D = -2r_0 \cos \Omega_0 \) and \( E = r_0^2 \). Solving, we find that

\[
\begin{align*}
    r_0 &= \sqrt{E} \\
    \Omega_0 &= \cos^{-1} \left( \frac{-D}{2r_0} \right) = \cos^{-1} \left( \frac{-D}{2\sqrt{E}} \right)
\end{align*}
\]

e. Determine

- the length of time required for the envelope \( r_0^n \) of \( h[n] \) to diminish by a factor of \( e \).
- the period of the oscillations (i.e., \( \frac{2\pi}{\Omega_0} \)) in \( h[n] \), and
- the number of periods of oscillation in \( h[n] \) before it diminishes by a factor of \( e \).

Express your results as functions of \( D \) and \( E \) only.
f. Estimate the parameters in part e for a DT system with the following poles:

\[ z\text{-plane} \]

\[ 0.149 \]

\[ 0.938 \]

From the plot \( \Omega_0 = \tan^{-1} \frac{0.149}{0.938} \approx 0.16 \) radians and \( r_0 = \sqrt{0.149^2 + 0.938^2} \approx 0.95 \).

The time to decay by a factor of \( e \) is \( \frac{-1}{\ln 0.95} \approx 19.5 \).

The period is \( \frac{2\pi}{\Omega_0} = \frac{2\pi}{0.16} \approx 39.3 \).

The number of cycles before decaying by \( e \) is \( \frac{19.5}{39.3} \approx 0.5 \).

2. Matches

The following plots show pole-zero diagrams, impulse responses, Bode magnitude plots, and Bode angle plots for six causal CT LTI systems. Determine which corresponds to which and fill in the following table.

**Pole-zero diagram 1** has a single pole at zero. The impulse response of a system with a single pole at zero is a unit step function (3). We evaluate the frequency response by considering frequencies along the \( j\omega \) axis. As we move away from the pole at the origin the log-magnitude decays linearly (5). The phase is constant since the angle between the pole and any point along positive side of the \( j\omega \) axis remains constant at \( \pi/2 \). The angle of the frequency response is therefore \( -\pi/2 \) (4).

**Pole-zero diagram 4** has a single pole at \( s = -1 \). The impulse response has the form \( e^{st}u(t) = e^{-t}u(t) \) (2). As we move along the \( j\omega \) axis, we move away from the pole at the origin, and the log-magnitude will eventually decay linearly. Because the pole is not
exactly at the origin, this decay is not significant until \( \omega = 1 \) (6). The phase starts at 0, and eventually moves to \(-\pi/2\). Note that as we move farther up the \( j\omega \) axis, this system behaves like the system of diagram 1 (2).

**Pole-zero diagram 3** adds a zero at the origin. A zero at the origin corresponds to taking the derivative, so we take the impulse response of pole-zero diagram 4 (2) and take its derivative (4). When \( \omega \) is small, the zero is dominant. As we move away from \( \omega = 0 \), the effect of the zero diminishes and the log-magnitude increases linearly. For sufficiently large \( \omega \) we are far enough that the zero and pole appear to cancel each other, and the magnitude becomes a constant (3). A zero at the origin means that we take the phase response of pole-zero diagram 4 (2) and add \( \pi/2 \) to it (6).

**Pole-zero diagram 2** contains complex conjugate poles
\[
H(s) = \frac{K}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)} = \frac{jA}{s + \sigma + j\omega_d} - \frac{jA}{s + \sigma - j\omega_d}.
\]

The impulse response has the form
\[
h(t) \propto e^{-\sigma t}(e^{j\omega_d t} - e^{-j\omega_d t}) \propto e^{-\sigma t} \sin \omega_d t
\]
which is response (1). The magnitude response will eventually decay twice as fast as that of pole-zero diagram 4 (6). Since there are two poles, there will be a bump at around \( \omega = 1 \) (2). At the origin, the angular contributions of the two poles cancel each other out, hence the angle is zero. As we move up the \( j\omega \) axis, the angles add up to \(-\pi\), with each pole contributing \(-\pi/2\) (3).

**Pole-zero diagram 6** adds a zero at the origin, meaning that we take the derivative of the impulse response of pole-zero diagram 2 (1). The derivative ends up being the combination of a decaying \( \cos(t) \) term minus a decaying \( \sin(t) \) term (5). The zero at the origin adds a linearly increasing component to the magnitude function (4). It also adds \( \pi/2 \) to the phase response everywhere (5).

**Pole-zero diagram 5** has complex conjugate poles and zeros at the same frequency \( \omega \). The system function has the form
\[
H(s) = \frac{s^2 - \frac{\omega_0^2}{Q}s + \omega_0^2}{s^2 + \frac{\omega_0^2}{Q}s + \omega_0^2}.
\]

This denominator has the same form as pole-zero diagrams 2 and 6, but has an additional power of \( s \) (corresponding to differentiation) in the numerator. This leads to a response of the form in (6). The symmetry of the poles and zeros means they cancel each other’s effect on magnitude (1). The phase response at \( \omega = 0 \) is zero, as the contributions cancel each other out. As we move past \( \omega = 1 \) where the conjugates are located, the phase moves in the negative direction faster, but eventually settles back at 0 as we move farther and the contributions again cancel each other out (1).

<table>
<thead>
<tr>
<th>Diagram</th>
<th>( h(t) )</th>
<th>Magnitude</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>PZ diagram 1</td>
<td>3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>PZ diagram 2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>PZ diagram 3</td>
<td>4</td>
<td>3</td>
<td>6</td>
</tr>
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<td>2</td>
<td>6</td>
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</tr>
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<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>PZ diagram 6</td>
<td>5</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
3. Desired oscillations

The following feedback circuit was the basis of Hewlett and Packard’s founding patent.

\[ V_{in} \rightarrow K \rightarrow R \rightarrow 1 \rightarrow R \rightarrow 1 \rightarrow R \rightarrow V_{out} \]

(a.) With \( R = 1 \, k\Omega \) and \( C = 1 \, \mu F \), sketch the pole locations as the gain \( K \) varies from 0 to \( \infty \), showing the scale for the real and imaginary axes. Find the \( K \) for which the system is barely stable and label your sketch with that information. What is the system’s oscillation period for this \( K \)?

The closed-loop gain is

\[ H(s) = \frac{K}{1 + (1 + sRC)^3} = \frac{K}{(1 + sRC)^3 + K} \]

The denominator is zero if

\[ (1 + sRC)^3 = -K \]

\[ (1 + sRC) = \sqrt[3]{-K} \]

\[ s = \frac{-1 + \sqrt[3]{-K}}{RC} \]

There are three cube roots of \(-K\): \(-\sqrt[3]{K}\), \(\sqrt[3]{K}e^{j\pi/3}\), and \(\sqrt[3]{K}e^{-j\pi/3}\) and three corresponding poles:

\[ s = \frac{-1 - \sqrt[3]{K}}{RC}, \frac{-1 + \sqrt[3]{K}e^{j\pi/3}}{RC}, \text{ and } \frac{-1 + \sqrt[3]{K}e^{-j\pi/3}}{RC} \]

The point of marginal stability is where the root locus crosses the \( j\omega \) axis. This occurs when the real part of \(-1 + \sqrt[3]{K}e^{j\pi/3}\) equals zero:

\[ \sqrt[3]{K} = 2 \]

so that \( K = 8 \). The frequency of oscillation is \( \omega = \frac{\sqrt{3}}{RC} \) so the period of oscillation is

\[ T = \frac{2\pi}{\omega} = \frac{2\pi RC}{\sqrt{3}} \].
For $RC = 1 \text{ ms}$ (as given), the period $T = 3.63 \text{ ms}$.

b. How do your results change if $R$ is increased to $10 \text{ k}\Omega$?

Increasing $R$ by a factor of 10 increases the period $T$ by a factor of 10, to $T = 36.3 \text{ ms}$. It has no effect of the critical value of $K = 8$.

4. Robotic steering

Design a steering controller for a car that is moving forward with constant velocity $V$.

You can control the steering-wheel angle $w(t)$, which causes the angle $\theta(t)$ of the car to change according to

$$\frac{d\theta(t)}{dt} = \frac{V}{d} w(t)$$

where $d$ is a constant with dimensions of length. As the car moves, the transverse position $p(t)$ of the car changes according to

$$\frac{dp(t)}{dt} = V \sin(\theta(t)) \approx V \theta(t) .$$

Consider three control schemes:

a. $w(t) = Ke(t)$

b. $w(t) = K_v \dot{e}(t)$

c. $w(t) = Ke(t) + K_v \dot{e}(t)$

where $e(t)$ represents the difference between the desired transverse position $x(t) = 0$ and the current transverse position $p(t)$. Describe the behaviors that result for each control scheme when the car starts with a non-zero angle ($\theta(0) = \theta_0$ and $p(0) = 0$). Determine the most acceptable value(s) of $K$ and/or $K_v$ for each control scheme or explain why none are acceptable.

**Part a.** This system can be represented by the following block diagram:

We are given a set of initial conditions — $p(0) = 0$ and $\theta(0) = \theta_0$ — and we are asked to characterize the response $p(t)$. Initial conditions are easy to take into account when a system is described by differential equations. However, feedback is easiest to analyze for systems expressed as operators or (equivalently) Laplace transforms. Therefore we first calculate the closed-loop system function,

$$H(s) \frac{Y(s)}{X(s)} = \frac{KV \frac{1}{d} V}{1 + K \frac{1}{d} V \frac{1}{s^2}} = \frac{K \frac{V^2}{d}}{s^2 + K \frac{V^2}{d}}$$
which has two poles: $\pm j\omega_0$ where $\omega_0 = V\sqrt{\frac{K}{d}}$. We can convert the system function to a differential equation:

$$\ddot{p}(t) + K\frac{V^2}{d}p(t) = K\frac{V^2}{d}x(t)$$

and then find the solution when $x(t) = 0$,

$$\ddot{p}(t) + K\frac{V^2}{d}p(t) = 0$$

so that $p(t) = C\sin\omega_0t$ since $p(0) = 0$.

From $p(t)$ we can calculate $\theta(t) = \dot{p}(t)/V = \frac{C}{V}\omega_0\cos\omega_0t$. From the initial condition $\theta(0) = \theta_0$, it follows that $C = V\theta_0/\omega_0$ and

$$p(t) = \frac{V\theta_0}{\omega_0}\sin\omega_0t = \theta_0\sqrt{\frac{d}{K}}\sin V\sqrt{\frac{K}{d}}t$$

for $t > 0$.

If $K$ is small, then the oscillations are slow, but they have a large amplitude. If $K$ is large, then the oscillations are fast (and therefore uncomfortable for passengers), but the amplitude is small. While none of these behaviors are desirable, it would probably be best to increase $K$ so that the amplitude of the oscillation is small enough so that the car stays in its lane.

**Part b.** The system can be represented by the following block diagram:

![Block Diagram](image)

The closed-loop system function is

$$H(s) = \frac{K_v s \frac{V^2}{d} \frac{1}{s^2}}{1 + K_v s \frac{V^2}{d} \frac{1}{s^2}} = \frac{K_v s \frac{V^2}{d}}{s(s + K_v \frac{V^2}{d})}.$$  

The closed-loop poles are at $s = 0$ and $s = -\frac{K_v}{d} V^2$.

Since $p(0) = 0$, the form of $p(t)$ is given by

$$p(0) = C\left(1 - e^{-\frac{K_v}{d} V^2 t}\right)$$

for $t > 0$. We can find $C$ by relating $C$ to the initial value of $\theta(t) = \dot{p}(t)/V$. Since $\theta(0) = \theta_0$, $\dot{p}(0) = V\theta_0$. Therefore $C = \frac{\theta_0}{K_v \frac{V}{d}}$, so that

$$p(t) = \frac{\theta_0}{K_v \frac{V}{d}}\left(1 - e^{-\frac{K_v}{d} V^2 t}\right)$$

for $t > 0$ as shown below.

![Graph](image)

We would like to make $K_v$ large because large $K_v$ leads to fast convergence. Large values of $K_v$ also lead to smaller steady-state errors in $p(t)$.

There are no oscillations in $p(t)$ with the velocity sensor, which is an advantage over results with the position sensor in part a. However, there is now a steady-state error in $p(t)$, which is worse. Fortunately the steady-state error can be made small with large $K_v$. 
Part c. The system can be represented by the following block diagram:

\[ X \xrightarrow{+} K_v s + K \xrightarrow{W V 1/2 \Theta} \frac{V}{s} \xrightarrow{P} \]

The closed-loop system function is

\[
H(s) = \frac{(K_v s + K) V^2}{s^2 + (K_v s + K) V^2 s} = \frac{(K_v s + K) V^2}{s^2 + \frac{1}{Q} s \omega_0 + \omega_0^2}
\]

This second-order system has a resonant frequency \( \omega_0 = \sqrt{\frac{K}{\pi} V^2} \) and a quality factor \( Q = \frac{K}{K_v \omega_0} \).

There is an enormous variety of acceptable solutions to this problem, since there are many values of \( K \) and \( K_v \) that can work. Here, we focus on one line of reasoning based on our normalization of second-order system in terms of \( Q \) and \( \omega_0 \).

To avoid excessive oscillations, we would like \( Q \) to be small. Try \( Q = 1 \). Then

\[
H(s) = \frac{(K_v s + K) V^2}{s^2 + \omega_0 s + \omega_0^2}.
\]

Then \( p(t) \) has the form

\[
p(t) = C e^{-\omega_0 t/2} \sin \left( \frac{\sqrt{3}}{2} \omega_0 t \right).
\]

As before, we can use the intial condition of \( \theta(0) = \theta_0 \) to determine \( C \). In general, \( \theta(t) = \dot{p}(t)/V \) so \( \ddot{p}(0) = \theta_0 V = C \sqrt{3} \omega_0 / 2 \). Therefore

\[
p(t) = \frac{2 V \theta_0}{\sqrt{3} \omega_0} e^{-\omega_0 t/2} \sin \left( \frac{\sqrt{3}}{2} \omega_0 t \right).
\]

Increasing \( Q \) would reduce the overshoot but slow the response. We could compensate for the slowig of the response by increasing \( \omega_0 \).

Performance can be adjusted to be better than either part a or part b. By adjusting \( Q \) and \( \omega_0 \) we can get convergence of \( p(t) \) to zero with minimum oscillation.

Although the steady-state value of the error is zero and the oscillation is minimized, there is still a transient behavior, which could momentarily move the car into the other lane!