Name: Kerberos
Username:

Please circle your section number:

<table>
<thead>
<tr>
<th>Section</th>
<th>Instructor</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Marc Baldo</td>
<td>10 am</td>
</tr>
<tr>
<td>2</td>
<td>Marc Baldo</td>
<td>11 am</td>
</tr>
<tr>
<td>3</td>
<td>Elfar Adalsteinsson</td>
<td>1 pm</td>
</tr>
<tr>
<td>4</td>
<td>Elfar Adalsteinsson</td>
<td>2 pm</td>
</tr>
</tbody>
</table>

Partial credit will be given for answers that demonstrate some but not all of the important conceptual issues.

Explanations are not required and will not affect your grade.

You have **three hours**.
Please put your initials on all subsequent sheets.
Enter your answers in the boxes.
This quiz is closed book, but you may use four 8.5 × 11 sheets of paper (eight sides total).
No calculators, computers, cell phones, music players, or other aids.

<table>
<thead>
<tr>
<th>1</th>
<th>/15</th>
</tr>
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<tbody>
<tr>
<td>2</td>
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<tr>
<td>3</td>
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<td>7</td>
<td>/16</td>
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<td>Total</td>
<td>/100</td>
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</table>
1. CT System with Feedback  [15 points]

Let $G$ represent a causal system that is described by the following differential equation

$$\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - x(t)$$

where $x(t)$ represents the input signal and $y(t)$ represents the output signal.

a. Determine the output $y_1(t)$ of $G$ when the input is

$$x_1(t) = \begin{cases} e^{-t}; & t \geq 0 \\ 0; & \text{otherwise} \end{cases}$$

Enter your result in the box below.

$$y_1(t) = (1 - 2t) e^{-t} u(t)$$

\[
\begin{align*}
(s + 1)Y &= (s - 1)X \\
Y &= \frac{s - 1}{s + 1} X \\
Y_1 &= \frac{s - 1}{s + 1} X_1; \quad X_1 = \frac{1}{s + 1}; \quad \text{Re} s > -1 \\
Y_1 &= \frac{s - 1}{(s + 1)^2} = \frac{1}{s + 1} - \frac{2}{(s + 1)^2} \\
y_1(t) &= e^{-t} u(t) - 2t e^{-t} u(t)
\end{align*}
\]
Now consider a feedback loop that contains the $G$ system described on the previous page.¹

\[
\begin{align*}
\begin{array}{c}
w(t) \quad + \\
K \quad \times \\
y(t)
\end{array}
\end{align*}
\]

b. Determine a differential equation that relates $w(t)$ to $y(t)$ when $K = 10$. The differential equation should not contain references to $x(t)$.

Enter the differential equation in the box below.

\[
11 \frac{dy(t)}{dt} - 9y(t) = 10 \frac{dw(t)}{dt} - 10w(t)
\]

\[
\begin{align*}
\frac{Y}{W} &= \frac{K \frac{s - 1}{s+1}}{1 + K \frac{s - 1}{s+1}} = K \frac{s - 1}{s+1 + Ks - K} = K \frac{s - 1}{(K+1)s - (K-1)} = K \frac{s - 1}{11s - 9}
\end{align*}
\]

¹ The minus sign near the adder indicates that the output of the adder is $w(t) - y(t)$
c. Determine the values of $K$ for which the feedback system on the previous page is stable. Enter the range (or ranges) in the box below.

\[ -1 < K < 1 \]

\[
\frac{Y}{W} = \frac{s - 1}{(K + 1)s - (K - 1)}
\]

\[ s = \frac{K - 1}{K + 1} < 0 \]

To be stable, the pole must be in the left half-plane.

Consider two cases: either $K$ is greater than $-1$ or it is less than $-1$.

If $K > -1$, then the denominator is positive and $K - 1$ must be less than 0 to assure that the pole is in the left half-plane. This implies $K < 1$. Thus $K$ could be in the range for $-1$ to $1$.

Now consider the possibility that $K$ is less than $-1$. Then $K - 1$ would have to be greater than 0 for the pole to be in the left half-plane. But this could only happen if $K > 1$, which contradicts the assumption that $K < -1$. Therefore $K$ cannot be less than $-1$. 

2. Stepping Up and Down  [10 points]

Use a small number of delays, gains, and 2-input adders (and no other types of elements) to implement a system whose response (starting at rest) to a unit-step signal

\[ x[n] = u[n] = \begin{cases} 
1 & n \geq 0 \\
0 & \text{otherwise} 
\end{cases} \]

is

\[ y[n] = \begin{cases} 
0 & n < 0 \\
1 & n = 0, 3, 6, 9, \ldots \\
2 & n = 1, 4, 7, 10, \ldots \\
3 & n = 2, 5, 8, 11, \ldots 
\end{cases} \]

Draw a block diagram of your system below.

First find a system whose unit-sample response is the desired sequence. The periodicity of 3 suggests that \( y[n] \) depends on \( y[n - 3] \). To get the correct numbers, just delay the input and weight the delays appropriately. The resulting difference equation is

\[ y[n] = y[n - 3] + w[n] + 2w[n - 1] + 3w[n - 2]. \]

A direct realization of the difference equation is shown below.

We can “reuse” 2 delays by commuting the left and right parts of this network.

Next compute \( w[n] \) which is the first difference of \( x[n] \):

\[ w[n] = x[n] - x[n - 1] \]

The result is the cascade of the first difference and the previous result.
3. **DT systems** \([14 \text{ points}]\)

The pole-zero diagram for a DT system is shown below, where the circle has radius 1.

![Pole-zero diagram](image)

It is known that when the input is 1 for all \(n\), the output is also 1 for all \(n\).

Sketch the unit-sample response \(h[n]\) of the system on the axes below. Label the important features of your sketch.

\[
Y \frac{X}{X} = K \frac{(z + \frac{1}{2} - j\frac{\sqrt{3}}{2})(z + \frac{1}{2} + j\frac{\sqrt{3}}{2})}{z} = K \frac{(z + \frac{1}{2})^2 + \frac{3}{4}}{z} = K \frac{z^2 + z + 1}{z} = K (z+1+z^{-1})
\]

\(H(1) = 3K = 1\)

Therefore, \(K = \frac{1}{3}\).

\(h[n] = \frac{1}{3} (\delta[n - 1] + \delta[n] + \delta[n + 1])\)
A second DT system has the following pole-zero diagram:

![Pole-Zero Diagram]

It is known that the system function $H(z)$ is 1 when $z = 1$.

Sketch the magnitude of the frequency response of this system on the axes below. Label the important features of your sketch (including the axes).

![Magnitude of Frequency Response]

Sketch the angle of the frequency response of this system on the axes below. Label the important features of your sketch (including the axes).

![Angle of Frequency Response]
4. CT Systems  [14 points]

A causal CT system has the following pole-zero diagram:

\[
\begin{array}{c}
\times \\
-1 - \frac{1}{2}
\end{array}
\]

Let \( y(t) = s(t) \) represent the response of this system to a unit-step signal

\[
x(t) = u(t) = \begin{cases} 
1; & t \geq 0 \\
0; & \text{otherwise}
\end{cases}
\]

Assume that the unit-step response \( s(t) \) of this system is known to approach 1 as \( t \to \infty \). Determine \( y(t) = s(t) \) and enter it in the box below.

\[
y(t) = (1 + e^{-t} - 2e^{-t/2}) u(t)
\]

From the pole-zero diagram,

\[
H(s) = \frac{K}{(s + 1)(s + \frac{1}{2})}.
\]

Since the system is stable (system is causal and poles are all in left half-plane), the unit-step response will approach \( H(0) \) as \( t \to \infty \). Therefore

\[
H(0) = \frac{K}{1(\frac{1}{2})} = 2K = 1
\]

and \( K = \frac{1}{2} \). The Laplace transform of the unit step is \( X(s) = \frac{1}{s} \) for \( \text{Re}\{s\} > 0 \). Therefore

\[
Y = \frac{\frac{1}{2}}{s(s + 1)(s + \frac{1}{2})} = \frac{1}{s} + \frac{1}{s + 1} - \frac{2}{s + \frac{1}{2}}
\]
A second CT system has the following pole-zero diagram:

- - - 1 s-plane
- - - 1
- - - -1

Assume that the input signal is

\[ x(t) = \begin{cases} 
1; & \text{cos} \, t > \frac{1}{\pi} \\
0; & \text{otherwise} 
\end{cases} \]

Let \( a_k \) and \( b_k \) represent the Fourier series coefficients of the input and output signals, respectively, where the fundamental (lowest frequency component) of each signal has a period of \( 2\pi \).

It is known that \( \frac{b_0}{a_0} = 1 \). Determine \( \frac{b_1}{a_1} \).

\[ \frac{b_1}{a_1} = \begin{bmatrix} 2 \\ -1 + 3j \end{bmatrix} \]

\[ H(s) = \frac{K}{(s + 1)(s + 1 - j\omega)(s + 1 + j\omega)} = \frac{K}{(s + 1)((s + 1)^2 + 1)} = \frac{K}{(s + 1)(s^2 + 2s + 2)} \]

\[ \frac{b_0}{a_0} = 1 = H(j0) = \frac{K}{2} \]

The fundamental frequency is \( \omega = \frac{2\pi}{2\pi} = 1 \).

\[ \frac{b_1}{a_1} = H(j1) = \frac{K}{(1 + j)(1 + 2j)} = \frac{2}{-1 + 3j} \]
5. DT processing of CT signals  [15 points]

Consider the following system for DT processing of CT signals:

\[ x_a(t) \xrightarrow{H_1(j\omega)} x_b(t) \xrightarrow{\text{uniform sampler}} x_c[n] \xrightarrow{H_2(e^{j\Omega})} y_d[n] \xrightarrow{\text{sample-to-impulse}} y_e(t) \xrightarrow{H_3(j\omega)} y_f(t) \]

where \( x_c[n] = x_b(nT) \) and

\[ y_e(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT). \]

The frequency responses \( H_1(j\omega) \) and \( H_3(j\omega) \) are given below.

\[ H_1(j\omega) \]

\[ H_3(j\omega) \]

a. Assume in this part that \( H_2(e^{j\Omega}) = 1 \) for all frequencies \( \Omega \). Determine \( y_f(t) \) when

\[ x_a(t) = \cos\left(\frac{\pi}{2T}t\right) + \sin\left(\frac{5\pi}{4T}t\right). \]

\[ y_f(t) = \cos\left(\frac{\pi}{2T}t\right) - \frac{1}{4} \sin\left(\frac{3\pi}{4T}t\right) \]
b. For this part, assume that

\[ H_2(e^{j\Omega}) = \begin{cases} 
1; & |\Omega| < \Omega_c \\
0; & \Omega_c < |\Omega| < \pi.
\end{cases} \]

For what values of \( \Omega_c \) is the overall system from \( x_a(t) \) to \( y_f(t) \) linear and time-invariant?

values of \( \Omega_c \): \[ 0 < \Omega_c < \frac{\pi}{2} \]
6. Which are True? [16 points]

For each of the DT signals $x_1[n]$ through $x_4[n]$ (below), determine whether the conditions listed in the following table are satisfied, and answer T for true or F for false.

<table>
<thead>
<tr>
<th>Condition</th>
<th>$x_1[n]$</th>
<th>$x_2[n]$</th>
<th>$x_3[n]$</th>
<th>$x_4[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(e^{j0}) = 0$</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$\int_{-\pi}^{\pi} X(e^{j\Omega}) d\Omega = 0$</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>$X(e^{j\Omega})$ is purely imaginary</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>$e^{jk\Omega}X(e^{j\Omega})$ is purely real for some integer $k$</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
7. Multiplied Sampling  [16 points]

The Fourier transform of a signal $x_a(t)$ is given below.

$$X_a(j\omega) \begin{cases} 1 & \frac{-\pi}{2} < \omega < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

This signal passes through the following system

$$x_a(t) \times x_b(t) \xrightarrow{\text{uniform sampler}} x_c[n] \xrightarrow{K} x_d[n] \xrightarrow{\text{sample-to-impulse}} x_e(t) \xrightarrow{H(j\omega)} x_f(t)$$

where $x_c[n] = x_b(nT)$ and

$$x_e(t) = \sum_{n=-\infty}^{\infty} x_d[n] \delta(t - nT)$$

and

$$H(j\omega) = \begin{cases} T & \text{if } |\omega| < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

a. Sketch the Fourier transform of $x_f(t)$ for the case when $K = 1$ and $T = 1$. Label the important features of your plot.
b. Is it possible to adjust $T$ and $K$ so that $x_f(t) = x_a(t)$?

**yes or no:** yes

If yes, specify a value $T$ and the corresponding value of $K$ (there may be multiple solutions, you need only specify one of them).

$T =$ \[
\begin{array}{c}
\frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{8}{7}, \frac{10}{7}, \frac{12}{7}, \text{ or } 2 \\
\end{array}
\]

$K =$ 1

If no, briefly explain why not.
c. Is it possible to adjust $T$ and $K$ so that the Fourier transform of $x_f(t)$ is equal to the following, and is zero outside the indicated range?

![Fourier Transform Diagram]

**yes or no:** yes

If yes, specify all possible pairs of $T$ and $K$ that work in the table below. If there are more rows in the table than are needed, leave the remaining entries blank.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$K$</th>
</tr>
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<tbody>
<tr>
<td>$\frac{1}{3}$</td>
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</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td>2</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>2</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>2</td>
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<td></td>
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</tbody>
</table>

If no, briefly explain why not.