Name:
Kerberos Username:

Please circle your section number:

<table>
<thead>
<tr>
<th>Section</th>
<th>Instructor</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Marc Baldo</td>
<td>10 am</td>
</tr>
<tr>
<td>2</td>
<td>Marc Baldo</td>
<td>11 am</td>
</tr>
<tr>
<td>3</td>
<td>Elfar Adalsteinsson</td>
<td>1 pm</td>
</tr>
<tr>
<td>4</td>
<td>Elfar Adalsteinsson</td>
<td>2 pm</td>
</tr>
</tbody>
</table>

Partial credit will be given for answers that demonstrate some but not all of the important conceptual issues.

Explanations are not required and will not affect your grade.

You have two hours.

Please put your initials on all subsequent sheets.

Enter your answers in the boxes.

This quiz is closed book, but you may use one 8.5 × 11 sheet of paper (two sides).

No calculators, computers, cell phones, music players, or other aids.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>/20</td>
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</tr>
<tr>
<td>2</td>
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<tr>
<td>3</td>
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<td>5</td>
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<td>Total</td>
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<td>/100</td>
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</tbody>
</table>
1. **Difference equation**  \[20\] points

Consider the system described by the following difference equation:

\[ y[n] = \alpha x[n] + \beta x[n - 1] - y[n - 2]. \]

**a.** Assume that the system starts at rest and that the input \( x[n] \) is the unit-step signal \( u[n] \).

\[ x[n] = u[n] \equiv \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

![Input signal](image)

Find \( y[119] \) and enter its value in the box below.

\[ y[119] = \boxed{0} \]

We can solve the difference equation by iterating, as shown in the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \alpha x[n] + \beta x[n - 1] )</th>
<th>( y[n] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>1</td>
<td>( \alpha + \beta )</td>
<td>( \alpha + \beta )</td>
</tr>
<tr>
<td>2</td>
<td>( \alpha + \beta )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>3</td>
<td>( \alpha + \beta )</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>( \alpha + \beta )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>5</td>
<td>( \alpha + \beta )</td>
<td>( \alpha + \beta )</td>
</tr>
<tr>
<td>6</td>
<td>( \alpha + \beta )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>7</td>
<td>( \alpha + \beta )</td>
<td>0</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( 4i )</td>
<td>( \alpha + \beta )</td>
<td>( \alpha )</td>
</tr>
<tr>
<td>( 4i + 1 )</td>
<td>( \alpha + \beta )</td>
<td>( \alpha + \beta )</td>
</tr>
<tr>
<td>( 4i + 2 )</td>
<td>( \alpha + \beta )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>( 4i + 3 )</td>
<td>( \alpha + \beta )</td>
<td>0</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

\[ y[119] = y[4 \times 29 + 3] = 0. \]
Consider the same system again.

\[ y[n] = \alpha x[n] + \beta x[n - 1] - y[n - 2] \]

**b.** Let \( \alpha = 3 \) and \( \beta = 4 \). Assume that the system starts at rest and that the input \( x[n] \) is the **unit-sample** signal.

\[
x[n] \quad 1 \quad n
\]

Determine coefficients \( A \) and \( B \) so that the response is

\[ A j^n + B(-j)^n ; \quad \text{for } n \geq 0. \]

Enter the coefficients in the boxes below, or enter **none** if no such coefficients can be found.

\[
A = \begin{bmatrix} \frac{3}{2} - j2 \\ \frac{3}{2} + j2 \end{bmatrix} \quad B = \begin{bmatrix} \frac{3}{2} + j2 \end{bmatrix}
\]

Express the difference equation as an operator expression:

\[
\frac{Y}{X} = \frac{3 + 4R}{1 + R^2} \times \frac{\frac{3}{2} - j2}{1 - jR} + \frac{\frac{3}{2} + j2}{1 + jR}
\]

The corresponding unit-sample response is

\[
\left(\frac{3}{2} - j2\right) j^n + \left(\frac{3}{2} + j2\right) (-j)^n ; \quad n \geq 0.
\]

Thus \( A = \frac{3}{2} - j2 \) and \( B = \frac{3}{2} + j2. \)
2. Feedback [20 points]

Consider the following system.

![System Diagram]

Assume that $X$ is the unit-sample signal, $x[n] = \delta[n]$. Determine the values of $\alpha$ and $\beta$ for which $y[n]$ is the following sequence (i.e., $y[0]$, $y[1]$, $y[2]$, ...):

$$0, 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots$$

Enter the values of $\alpha$ and $\beta$ in the boxes below. Enter none if the value cannot be determined from the information provided.

$$\alpha = \frac{1}{2}, \quad \beta = 2$$

Express the block diagram as a system functional:

$$Y = \frac{\alpha R}{1 - \frac{3}{2}R} \beta = \frac{\alpha \beta R}{1 - \frac{3}{2}R + \alpha R^2}.$$ 

The poles are at

$$3 \pm \sqrt{\frac{9}{16} - \alpha}.$$ 

Now express $y[n]$ as a weighted sum of geometrics:

$$y[n] = 2 \times 1^n - 2 \times \left(\frac{1}{2}\right)^n; \quad n \geq 0$$

Thus the poles must be at $z = 1$ and $z = \frac{1}{2}$. It follows that $\alpha$ must be $\frac{1}{2}$. Then the system functional is

$$Y = \frac{\frac{1}{2} \beta R}{1 - \frac{3}{2}R + \frac{1}{2}R^2} = \frac{\beta}{1 - \frac{3}{2}R} - \frac{\beta}{1 - \frac{1}{2}R}$$

and $\beta$ must be 2.
3. **Scaling time**  
*[20 points]*

A system containing only adders, gains, and delays was designed with system functional

\[ H = \frac{Y}{X} \]

which is a ratio of two polynomials in \( \mathcal{R} \). When this system was constructed, users were dissatisfied with its responses. Engineers then designed three new systems, each based on a different idea for how to modify \( H \) to improve the responses.

- **System** \( H_1 \): every delay element in \( H \) is replaced by a cascade of two delay elements.
- **System** \( H_2 \): every delay element in \( H \) is replaced by a gain of \( \frac{1}{2} \) followed by a delay.
- **System** \( H_3 \): every delay element in \( H \) is replaced by a cascade of three delay elements.

For each of the following parts, evaluate the truth of the associated statement and enter **yes** if the statement is always true or **no** otherwise.

**a.** If \( H \) has a pole at \( z = j = \sqrt{-1} \), then \( H_1 \) has a pole at \( z = e^{j5\pi/4} \).

Statement is always true (**yes** or **no**): **yes**

The poles of \( H \) are the roots of the denominator of \( H|_{\mathcal{R} \to \frac{1}{2}} \). But \( H_1 = H|_{\mathcal{R} \to \mathcal{R}^2} \). Thus the poles of \( H_1 \) are the roots of the denominator of \( H_1|_{\mathcal{R} \to \frac{1}{2}} = (H|_{\mathcal{R} \to \mathcal{R}^2})|_{\mathcal{R} \to \frac{1}{2}} = H|_{\mathcal{R} \to \frac{1}{4}} \). It follows that the poles of \( H_1 \) are the square roots of the poles of \( H \).

If \( H \) has a pole at \( z = j \) then \( H_1 \) must have poles at \( z = \pm \sqrt{j} \). The two square roots of \( j \) are \( e^{j\pi/4} \) and \( e^{j3\pi/4} \). Thus \( e^{j5\pi/4} \) is a pole of \( H_1 \).
b. If \( H \) has a pole at \( z = p \) then \( H_2 \) has a pole at \( z = 2p \).

Statement is always true (yes or no): no

The poles of \( H \) are the roots of the denominator of \( H|_{R \rightarrow \frac{1}{2}} \). But \( H_2 = H|_{R \rightarrow \frac{R}{2}} \). Thus the poles of \( H_2 \) are the roots of the denominator of \( H_2|_{R \rightarrow \frac{1}{2}} = \left( H|_{R \rightarrow \frac{R}{2}} \right)|_{R \rightarrow \frac{1}{2}} = H|_{R \rightarrow \frac{1}{4}} \). It follows that the poles of \( H_2 \) are half those of \( H \).

If \( H \) has a pole at \( z = p \) then \( H_2 \) must have poles at \( z = p/2 \) (not \( 2p \)).

c. If \( H \) is stable then \( H_3 \) is also stable (where a system is said to be stable if all of its poles are inside the unit circle).

Statement is always true (yes or no): yes

The poles of \( H \) are the roots of the denominator of \( H|_{R \rightarrow \frac{1}{2}} \). But \( H_3 = H|_{R \rightarrow \frac{R}{3}} \). Thus the poles of \( H_3 \) are the roots of the denominator of \( H_3|_{R \rightarrow \frac{1}{2}} = \left( H|_{R \rightarrow \frac{R}{3}} \right)|_{R \rightarrow \frac{1}{2}} = H|_{R \rightarrow \frac{1}{6}} \). It follows that the poles of \( H_3 \) are the cube roots of the poles of \( H \).

If \( H \) is stable, then the magnitudes of all of its poles are less than 1. It follows that the magnitudes of all of the poles of \( H_3 \) are also less than 1 since the magnitude of the cube root of a number that is less than 1 is also less than 1. Thus \( H_3 \) must also be stable.
4. Mystery Feedback  [20 points]

Consider the following feedback system where $F$ is the system functional for a system composed of just adders, gains, and delay elements.

\[ X \xrightarrow{+} \alpha \xrightarrow{} F \xrightarrow{} Y \]

If $\alpha = 10$ then the closed-loop system functional is known to be

\[ \left. \frac{Y}{X} \right|_{\alpha=10} = \frac{1 + \mathcal{R}}{2 + \mathcal{R}} \]

Determine the closed-loop system functional when $\alpha = 20$.

\[ \left. \frac{Y}{X} \right|_{\alpha=20} = \frac{2 + 2\mathcal{R}}{3 + 2\mathcal{R}} \]

In general

\[ \left. \frac{Y}{X} \right| = \frac{\alpha F}{1 + \alpha F} \]

If $\alpha = 10$

\[ \left. \frac{Y}{X} \right|_{\alpha=10} = \frac{10F}{1 + 10F} = \frac{1 + \mathcal{R}}{2 + \mathcal{R}} \]

We can solve for $F$ by equating the reciprocals of these expressions,

\[ \frac{1}{10F} + 1 = \frac{2 + \mathcal{R}}{1 + \mathcal{R}} \]

\[ \frac{1}{10F} = \frac{2 + \mathcal{R}}{1 + \mathcal{R}} - 1 = \frac{1}{1 + \mathcal{R}} \]

from which it follows that $10F = 1 + \mathcal{R}$. Then if $\alpha = 20$,

\[ \left. \frac{Y}{X} \right|_{\alpha=20} = \frac{20F}{1 + 20F} = \frac{2 + 2\mathcal{R}}{3 + 2\mathcal{R}} \]
5. **Ups and Downs**  

[20 points]

Use a small number of delays, gains, and 2-input adders (and no other types of elements) to implement a system whose unit-sample response \( (h[0], h[1], h[2], \ldots) \) (starting at rest) is

\[
1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots
\]

Draw a block diagram of your system below.

![Block diagram of a system with delays, gains, and adders](image)

Derive the difference equation. The periodicity of 3 suggests that \( y[n] \) depends on \( y[n-3] \). To get the correct numbers, just delay the input and weight the delays appropriately. The resulting difference equation is

\[
y[n] = y[n-3] + x[n] + 2x[n-1] + 3x[n-2].
\]

A direct realization of the difference equation is shown below.

![Direct realization of the difference equation](image)

We can “reuse” 2 delays by commuting the left and right parts of this network, which gives the answer above.