6.003: Signals and Systems

Fourier Series

November 1, 2011
Harmonic content is a natural way to describe some kinds of signals. For example, musical instruments (http://theremin.music.uiowa.edu/MIS.html).
Last Time: Fourier Series

Determining harmonic components of a periodic signal.

\[ a_k = \frac{1}{T} \int_T x(t) e^{-j \frac{2\pi}{T} kt} \, dt \]  

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j \frac{2\pi}{T} kt} \]  

(“analysis” equation)  

(“synthesis” equation)

We can think of Fourier series as an **orthogonal decomposition**.
Orthogonal Decompositions

**Vector representation of 3-space:** let \( \vec{r} \) represent a vector with components \( \{x, y, z\} \) in the \( \{\hat{x}, \hat{y}, \hat{z}\} \) directions, respectively.

\[
\begin{align*}
x &= \vec{r} \cdot \hat{x} \\
y &= \vec{r} \cdot \hat{y} \\
z &= \vec{r} \cdot \hat{z}
\end{align*}
\]

(“analysis” equations)

\[
\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}
\]

(“synthesis” equation)

**Fourier series:** let \( x(t) \) represent a signal with harmonic components \( \{a_0, a_1, \ldots, a_k\} \) for harmonics \( \{e^{j0t}, e^{j\frac{2\pi}{T}t}, \ldots, e^{j\frac{2\pi}{T}kt}\} \) respectively.

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a_k &= \frac{1}{T} \int_{T} x(t) e^{-j\frac{2\pi}{T}kt} dt \\
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Orthogonal Decompositions

**Vector representation of 3-space:** let $\vec{r}$ represent a vector with components $\{x, y, \text{and } z\}$ in the $\{\hat{x}, \hat{y}, \text{and } \hat{z}\}$ directions, respectively.

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(“analysis” equation)

(“synthesis” equation)
Orthogonal Decompositions

Integrating over a period sifts out the \( k^{\text{th}} \) component of the series.

Sifting as a dot product:
\[
x = \bar{r} \cdot \hat{x} \equiv |\bar{r}| |\hat{x}| \cos \theta
\]

Sifting as an inner product:
\[
a_k = e^{j\frac{2\pi}{T} kt} \cdot x(t) \equiv \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T} kt} dt
\]
where
\[
a(t) \cdot b(t) = \frac{1}{T} \int_T a^*(t)b(t)dt.
\]

The complex conjugate (*) makes the inner product of the \( k^{\text{th}} \) and \( m^{\text{th}} \) components equal to 1 iff \( k = m \):
\[
1 \frac{1}{T} \int_T (e^{j\frac{2\pi}{T} kt})^* (e^{j\frac{2\pi}{T} mt}) dt = \frac{1}{T} \int_T e^{-j\frac{2\pi}{T} kt} e^{j\frac{2\pi}{T} mt} dt = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}
\]
Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$?

1. $\cos 2\pi t \perp \sin 2\pi t$?
2. $\cos 2\pi t \perp \cos 4\pi t$?
3. $\cos 2\pi t \perp \sin \pi t$?
4. $\cos 2\pi t \perp e^{j2\pi t}$?
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp \sin 2\pi t$?

$\cos 2\pi t \perp \sin 2\pi t$?

$\cos 2\pi t \sin 2\pi t = \frac{1}{2} \sin 4\pi t$

$\int_{0}^{3} dt = 0$ therefore YES
Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \cos 4\pi t$?

$\int_0^3 dt = 0$ therefore YES
Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \sin \pi t$?

$\cos 2\pi t \perp \sin \pi t$?

$\int_{0}^{3} dt \neq 0$ therefore NO
How many of the following are orthogonal \((\perp)\) in \(T = 3\)?

\[
\cos 2\pi t \perp e^{2\pi t} \quad ?
\]

\[
e^{2\pi t} = \cos 2\pi t + j \sin 2\pi t
\]

\[
\cos 2\pi t \perp \sin 2\pi t \quad \text{but not} \quad \cos 2\pi t
\]

Therefore \textbf{NO}
Check Yourself

How many of the following pairs of functions are orthogonal (⊥) in $T = 3$? 2

1. $\cos 2\pi t \perp \sin 2\pi t$ ? √
2. $\cos 2\pi t \perp \cos 4\pi t$ ? √
3. $\cos 2\pi t \perp \sin \pi t$ ? ×
4. $\cos 2\pi t \perp e^{j2\pi t}$ ? ×
Vowel sounds are quasi-periodic.
Speech

Harmonic content is natural way to describe vowel sounds.
Speech

Harmonic content is a natural way to describe vowel sounds.
Speech Production

Speech is generated by the passage of air from the lungs, through the vocal cords, mouth, and nasal cavity.
Speech Production

Controlled by complicated muscles, vocal cords are set in vibration by the passage of air from the lungs.

Looking down the throat:

Gray's Anatomy

Adapted from T.F. Weiss
Speech Production

Vibrations of the vocal cords are “filtered” by the mouth and nasal cavities to generate speech.
Filtering

Notion of a filter.

LTI systems
• cannot create new frequencies.
• can only scale magnitudes & shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit

\[ \begin{align*}
R & \quad + \\
C & \quad v_i \\
\frac{1}{RC} & \quad - \\
\frac{1}{RC} & \quad + \\
v_o & \quad - \\
\end{align*} \]
Lowpass Filter

Calculate the frequency response of an RC circuit.

\[ v_i(t) = R i(t) + v_o(t) \]
\[ i(t) = C \dot{v}_o(t) \]

Solving:
\[ v_i(t) = R C \dot{v}_o(t) + v_o(t) \]
\[ V_i(s) = (1 + s R C) V_o(s) \]
\[ H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + s R C} \]
Lowpass Filtering

Let the input be a square wave.

\[ x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt}; \quad \omega_0 = \frac{2\pi}{T} \]
Lowpass Filtering

Low frequency square wave: \( \omega_0 << 1/RC \).

\[
x(t) = \sum_{k \text{ odd}} \frac{1}{j \pi k} e^{j \omega_0 k t} ; \quad \omega_0 = \frac{2\pi}{T}
\]

\[
|H(j\omega)| = \frac{1}{1 + (\omega / \omega_0)^2}
\]

\[
\angle H(j\omega) = -\frac{\pi}{2} - \pi \log_2 \left( \frac{\omega}{\omega_0} \right)
\]
Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt}; \quad \omega_0 = \frac{2\pi}{T}$$
Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.

$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt}; \quad \omega_0 = \frac{2\pi}{T}$$
Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.

$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt}; \quad \omega_0 = \frac{2\pi}{T}$$

Magnitude and phase response of the lowpass filter.
Source-Filter Model of Speech Production

Vibrations of the vocal cords are “filtered” by the mouth and nasal cavities to generate speech.

buzz from vocal cords -- throat and nasal cavities --> speech
Speech Production

X-ray movie showing speech in production.

Courtesy of Kenneth N. Stevens. Used with permission.
Demonstration

Artificial speech.

buzz from vocal cords

$H(j\omega)$

$y(t)$

throat and nasal cavities

speech
Formants

Resonant frequencies of the vocal tract.

![Formant Diagram]

<table>
<thead>
<tr>
<th></th>
<th>F1</th>
<th>F2</th>
<th>F3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>270</td>
<td>2290</td>
<td>3010</td>
</tr>
<tr>
<td>Women</td>
<td>310</td>
<td>2790</td>
<td>3310</td>
</tr>
<tr>
<td>Children</td>
<td>370</td>
<td>3200</td>
<td>3730</td>
</tr>
</tbody>
</table>

http://www.sfu.ca/sonic-studio/handbook/Formant.html
Speech Production

Same glottis signal + different formants → different vowels.

We detect changes in the filter function to recognize vowels.
Singing

We detect changes in the filter function to recognize vowels ... at least sometimes.

Demonstration.

“la” scale.

“lore” scale.

“loo” scale.

“ler” scale.

“lee” scale.

Low Frequency: “la” “lore” “loo” “ler” “lee”.

High Frequency: “la” “lore” “loo” “ler” “lee”.

We detect changes in the filter function to recognize vowels.

\[ |H_{ee}(j\omega)| \]

- **low**
- **intermediate**
- **high**
We detect changes in the filter function to recognize vowels.
Continuous-Time Fourier Series: Summary

Fourier series represent signals by their frequency content.

Representing a signal by its frequency content is useful for many signals, e.g., music.

Fourier series motivate a new representation of a system as a filter.

Representing a system as a filter is useful for many systems, e.g., speech synthesis.