how to design a SAT solver, part 2

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plan for today

topics

- designing a naive solver
- more recursive functions over datatypes

today’s patterns

- **Interpreter**: recursive traversals (again)
- **Backtracking Search**
- **Facade** for simpler use of API
where we are
datatypetype productions

last time we saw

• how to model formulas using datatype productions
• like a grammar, but abstract structure only

productions

Formula = OrFormula + AndFormula + Not(formula:Formula) + Var(name:String)
OrFormula = OrVar(left:Formula, right:Formula)
AndFormula = And(left:Formula, right:Formula)

sample formula: \((P \lor Q) \land (\neg P \lor R)\)
• as a term:
  \(\text{And(Or(Var(“P”), Var(“Q”)), (Not(Var(“P”)), Var(“R”)))}\)
Variant as Class pattern

last time we saw

‣ how to define a datatype to model a set of values
‣ how to build a class structure representing it
‣ how to implement recursive functions over the datatype

eample

‣ production

\[
\text{List}\langle\text{E}\rangle = \text{Empty} + \text{Cons (first: E, rest: List}\langle\text{E}\rangle)\\
\]

‣ code

```java
public abstract class List\langle E\rangle {}
public class Empty\langle E\rangle extends List\langle E\rangle {}
public class Cons\langle E\rangle extends List\langle E\rangle {
    private final E first;
    private final List\langle E\rangle rest;
    public Cons (E e, List\langle E\rangle r) {first = e; rest = r;}
    public E first () {return first;}
    public List\langle E\rangle rest () {return rest;}
}
```

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Interpreter pattern

how to build a recursive traversal

• write type declaration of function
  \[\text{size: List}\langle E\rangle \rightarrow \text{int}\]

• break function into cases, one per variant
  \[\text{List}\langle E\rangle = \text{Empty} + \text{Cons(first:}\ E, \text{rest: List}\langle E\rangle)\]
  \[\text{size (Empty)} = 0\]
  \[\text{size (Cons(first:}\ e, \text{rest: l)}) = 1 + \text{size(rest)}\]

• implement with one subclass method per case

```java
public abstract class List<E> {
    public abstract int size();
}

public class Empty<E> extends List<E> {
    public int size() { return 0; }
}

public class Cons<E> extends List<E> {
    private final E first;
    private final List<E> rest;
    public int size() { return 1 + rest.size(); }
}
```

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SAT solver functions
functions for SAT

generate and test strategy

• steps
  - extract set of variables from formula
  - try all environments over those vars
  - evaluate the formula for each

• functions
  - vars: Formula -> Set<Var>
  - solve: Formula -> Option<Env>
  - eval: Formula, Env -> Bool
what are the Set and Env types?

\[ Set<T> = List<T> \]
\[ Env = List<Tuple<Var, Boolean>> \]
\[ Boolean = True + False \]

something new going on here

\[ Set<T> = List<T> \]?

representation (on right) is hidden from clients

not all terms are acceptable: no duplicates, eg

more on this later when we discuss abstract types
set and env specs

assume for now

• Set and Env implemented as classes, with list representations
• but offering special methods:

```java
public class Set<E> {
    public Set () {...}
    public Set<E> add (E e) {...}
    public Set<E> remove (E e) {...}
    public Set<E> addAll (Set<E> s) {...}
    public boolean contains (E e) {...}
    public E choose () {...}
    public boolean isEmpty () {...}
    public int size () {...}
}

public class Env {
    public Env() {...}
    public Env put(Var v, boolean b) {...}
    public boolean get(Var v) {...} // requires: v is bound in this environment
    public boolean
```
computing var set

applying strategy

• write type declaration of function
  \[\text{vars}: \text{Formula} \rightarrow \text{Set<Var>}\]

• break function into cases, one per variant
  \[F = \text{Var(name:String)} + \text{Or(left:F, right:F)} + \text{And(left:F, right:F)} + \text{Not(formula:F)}\]
  \[\text{vars(Var(n))} = \{\text{Var(n)}\}\]
  \[\text{vars(Or(fl, fr))} = \text{vars(fl)} \cup \text{vars(fr)}\]
  \[\text{vars(And(fl, fr))} = \text{vars(fl)} \cup \text{vars(fr)}\]
  \[\text{vars(Not(f))} = \text{vars(f)}\]

• implement with one subclass method per case, eg

```java
public class AndFormula extends Formula {
    private final Formula left, right;
    public Set<Var> vars () {
        return left.vars().addAll(right.vars());
    }
}
```

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public abstract class Formula {
    public abstract Set<Var> vars();
}

public class AndFormula extends Formula {
    private final Formula left, right;
    public Set<Var> vars () {
        return left.vars().addAll(right.vars());
    }
}

public class OrFormula extends Formula {
    private final Formula left, right;
    public Set<Var> vars () {
        return left.vars().addAll(right(vars()));
    }
}

public class NotFormula extends Formula {
    private final Formula formula;
    public Set<Var> vars () {
        return formula.vars();
    }
}

public class Var extends Formula {
    public Set<Var> vars () {
        return new ListSet<Var>().add(this);
    }
}
in-class exercise

apply the strategy for eval

• write type declaration of function
  
  eval: Formula, Env -> Boolean

• break function into cases, one per variant
  
  F = Var(name:String) + Or(left:F,right:F) + And(left:F,right:F) + Not(formula:F)
  
  eval (Var(n), e) = e.get(Var(n))
  
  eval (Or(fl, fr), e) = eval(fl,e) || evals(fr,e)
  
  eval (And(fl, fr), e) = eval(fl,e) && eval(fr,e)
  
  eval (Not(f), e) = ! eval(f,e)

• implement with one subclass method per case, eg
  
  public class AndFormula extends Formula {
    private final Formula left, right;
    public boolean eval (Env e) {
      return left.eval (e) && right.eval (e);
    }
  }
public abstract class Formula {
    public abstract boolean eval (Env e);
}
public class AndFormula extends Formula {
    private final Formula left, right;
    public boolean eval (Env e) {
        return left.eval (e) && right.eval (e);
    }
}
public class OrFormula extends Formula {
    private final Formula left, right;
    public boolean eval(Env e) {
        return left.eval(e) || right.eval(e);
    }
}
public class NotFormula extends Formula {
    private final Formula formula;
    public boolean eval (Env e) {
        return !formula.eval (e);
    }
}
public class Var extends Formula {
    public boolean eval (Env e) {
        return e.get(this);
    }
}
a naive solver
naive SAT

backtracking search

• pick a var, and try setting to false and then to true if that fails
• do this recursively, evaluating the formula when no vars left

implementation

```java
public abstract class Formula {
    ...

    public Env solve () {
        return solve (new Env (), this.vars());
    }

    private Env solve(Env env, Set<Var> vars) {
        if (vars.isEmpty())
            return eval(env) ? env : null;
        Var v = vars.choose();
        Set<Var> restVars = vars.remove(v);
        Env e = solve (env.put(v, false), restVars);
        if (e != null) return e;
        return solve (env.put(v, true), restVars);
    }
}
```

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example

• formula \( f = \)

\[
Socrates \Rightarrow \text{Human} \land \text{Human} \Rightarrow \text{Mortal} \land \neg (Socrates \Rightarrow \text{Mortal})
\]

• \( \text{vars}(f) = \)

\[\{\text{Socrates, Human, Mortal}\}\]

• possible environments

\[\{\text{Socrates}\rightarrow\text{False, Human}\rightarrow\text{False, Mortal}\rightarrow\text{False}\}\]
\[\{\text{Socrates}\rightarrow\text{False, Human}\rightarrow\text{False, Mortal}\rightarrow\text{True}\}\]
\[\{\text{Socrates}\rightarrow\text{False, Human}\rightarrow\text{True, Mortal}\rightarrow\text{False}\}\]
\[\{\text{Socrates}\rightarrow\text{False, Human}\rightarrow\text{True, Mortal}\rightarrow\text{True}\}\]
\[\{\text{Socrates}\rightarrow\text{True, Human}\rightarrow\text{False, Mortal}\rightarrow\text{False}\}\]
\[\{\text{Socrates}\rightarrow\text{True, Human}\rightarrow\text{False, Mortal}\rightarrow\text{True}\}\]
\[\{\text{Socrates}\rightarrow\text{True, Human}\rightarrow\text{True, Mortal}\rightarrow\text{False}\}\]
\[\{\text{Socrates}\rightarrow\text{True, Human}\rightarrow\text{True, Mortal}\rightarrow\text{True}\}\]

• formula evaluates to false on all, so theorem holds
class exercise

what order are environments checked in?

depends on behaviour of Set.choose

assume it returns vars in this order

Socrates, Human, Mortal
running the example

```java
public static void main (String[] args) {
    Var s = new Var("Socrates");
    Var h = new Var("Human");
    Var m = new Var("Mortal");
    Formula old_f =
        new AndFormula(new OrFormula(new NotFormula(s), h),
                        new AndFormula(new OrFormula(new NotFormula(h), m),
                                       new NotFormula(new OrFormula(new NotFormula(s), m))));
    Environment e = f.solve();
    System.out.println("Solution: "+(e==null?"none":e));
}
```

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long started = System.nanoTime();
Sudoku s = new Sudoku (2);
System.out.println ("Creating SAT formula...");
Formula f = s.getFormula();
System.out.println ("Solving with naive method...");
Environment e = f.solve();
System.out.println ("Interpreting solution...");
String solution = s.interpretSolution(e);
System.out.println ("Solution is: \n" + solution);
long time = System.nanoTime();
long timeTaken = (time - started);
System.out.println ("Time:" + timeTaken/1000000 + "ms");

Creating SAT formula...
Solving with naive method...
Interpreting solution...
Solution is:
| 3 | 4 | 2 | 1 |
| 1 | 2 | 4 | 3 |
| 4 | 3 | 1 | 2 |
| 2 | 1 | 3 | 4 |
Time: 797ms
design extras
an awkward API

look at how formula is created by client

- tedious to have to use constructors and multiple classes

```
Formula f =
    new AndFormula (new OrFormula (new NotFormula (s), h),
        new AndFormula (new OrFormula (new NotFormula (h), m),
            new NotFormula (new OrFormula (new NotFormula (s), m))));
```

define methods in Formula class to avoid this: example of Facade

```
public abstract class Formula {
    public Formula and (Formula f) {
        return new AndFormula (this, f);
    }
    public Formula or (Formula f) {
        return new OrFormula (this, f);
    }
    public Formula not () {
        return new NotFormula (this);
    }
}
```

- can now write

```
Formula f = s.not().or(h).and(h.not().or(m).and(s.not().or(m).not()));
```
module dependency diagram
handling unbound vars

how should get method handle unbound var?

• one approach: return an arbitrary value
• technically correct, but not very robust

```java
public class Environment {
    Map <Var, Boolean> bindings;
    ...

    /**
     * requires that v is bound in this environment
     * @return the boolean value that v is bound to
     */
    public boolean get(Var v){
        Boolean b = bindings.get(v);
        if (b==null) return false;
        else return b;
    }
}
```
three-valued logic

an alternative: define 3 logical values

Boolean = True + False + Undefined

```java
public enum Bool {
    TRUE, FALSE, UNDEFINED;

    public Bool and (Bool b) {
        if (this==FALSE || b==FALSE) return FALSE;
        if (this==TRUE && b==TRUE) return TRUE;
        return UNDEFINED;
    }
    ...
}
```

now we can return undefined

```java
/**
 * @return the boolean value that v is bound to, or
 * the special UNDEFINED value of it is not bound
 */
public Bool get(Var v){
    Bool b = bindings.get(v);
    if (b==null) return Bool.UNDEFINED;
    else return b;
}
```
using Bool

use methods of Bool instead of &&, ||, etc

```java
public class AndFormula extends Formula {
    public Bool eval (Environment e) {
        return left.eval(e).and (right.eval (e));
    }
}
```

and in solver, can evaluate before all vars are bound

```java
public Environment solve () {
    return solve (new Environment (), this.vars());
}
```

```java
private Environment solve(Environment env, Set<Var> vars) {
    if (eval(env) == Bool.TRUE) return env;
    if (eval(env) == Bool.FALSE) return null;
    Var v = vars.choose();
    Set<Var> restVars = vars.remove(v);
    Environment e = solve (env.put(v, Bool.FALSE), restVars);
    if (e != null) return e;
    return solve (env.put(v, Bool.TRUE), restVars);
}
```

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puzzle

introduction of Bool

• produces dramatic performance improvement
• 4x4 Latin square actually doesn't terminate without it
• what's going on?
return type of solve

recall solve function

• prototype is
  
  solve: Formula -> Option<Env>

• recall option datatype
  
  Option<T> = Some(value:T) + None

how should this be implemented?

• we used nulls

• is there a better way?
public class Option<T> {}  
public class None<T> extends Option<T>{}
public class Some<T> extends Option<T>{
    private T value;
    public Some (T v) {value = v;}
    public T getValue () {return value;}
}

public void displaySolution () {
    Option<Environment> o = solve (new Environment (), this.vars());
    if (o instanceof Some)
        System.out.println (((Some<Environment>) o).getValue());
    else System.out.println ("No solution");
}

private Option<Environment> solve (Environment env, Set<Literal> vars) {
    if (eval(env) == Bool.TRUE) return new Some<Environment>(env);
    if (eval(env) == Bool.FALSE) return new None<Environment>();
    Var v = vars.choose();
    Set<Var> restVars = vars.remove(v);
    Option<Environment> o = solve (env.put (c, Bool.FALSE), restVars);
    if (o instanceof Some) return o;
    return solve (env.put(v, Bool.TRUE), restVars);
}
comparing options

two options for Option

- have solve return an Env or a null value
- implement Option<T> directly

others?

- throw an exception if not successful
- have solve return a pair (boolean, env)

class discussion

- advantages and disadvantages of each
abstract classes vs. interfaces
what’s an abstract class?

like a regular class
  but can’t be instantiated

like an interface
  but can contain fields and method bodies
  methods not implemented are marked abstract

why useful?
  can collect fields and methods common across subclasses
    eg: Formula.solve
  can use as Facade
    eg: Formula.and, Formula.or, Formula.not
using interfaces instead

changes to List

' code is now

```java
public interface List<E> {}
public class Empty<E> implements List<E> {}
public class Cons<E> implements List<E> {
    private final E first;
    private final List<E> rest;
    public Cons (E e, List<E> r) {first = e; rest = r;}
    public E first () {return first;}
    public List<E> rest () {return rest;}
}
```
public abstract class List<E> {
    int size;
    public int size () {return size;}
}

public class Empty<E> extends List<E> {
    public EmptyList () {size = 0;}
}

public class Cons<E> extends List<E> {
    private final E first;
    private final List<E> rest;
    private Cons (E e, List<E> r) {first = e; rest = r; size = r.size()+1}
}
and what becomes of this?

```java
public abstract class Formula {
    public Environment solve (Formula f) {
        return ...;
    }
    public Formula and (Formula f) {
        return new AndFormula (this, f);
    }
    public Formula or (Formula f) {
        return new OrFormula (this, f);
    }
    public Formula not () {
        return new NotFormula (this);
    }
}
```
public class Formulas {
    public static Environment solve (Formula f) {
        return ...;
    }
    public static Formula and (Formula f, Formula g) {
        return new AndFormula (f, g);
    }
    public static Formula or (Formula f, Formula g) {
        return new OrFormula (f, g);
    }
    public static Formula not (Formula f) {
        return new NotFormula (f);
    }
}
interfaces vs. abstract classes

advantages of interfaces
  • you know at compile time which method is executed
  • enforces clean specification

disadvantages
  • need extra (singleton) class for facade
  • can’t share code
what's wrong with our solver?
a missed opportunity

look at what happens

 › after

\[
\text{Socrates} \Rightarrow \text{Human} \land \text{Human} \Rightarrow \text{Mortal} \land \neg (\text{Socrates} \Rightarrow \text{Mortal})
\]

 › suppose order or evaluation is \text{Socrates, Human, Mortal}

 › and suppose we set \text{Socrates} to true

 › then clearly must set \text{Human} to true

 › and then must set \text{Mortal} to true...

 › but our solver ignores all this

next time

 › a real SAT solver

 › implements this scheme with unit propagation
summary
summary

big ideas

‣ backtracking search: easy with immutable types

patterns

‣ Variant as Class: abstract class for datatype, one subclass/variant
‣ Interpreter: recursive traversal over datatype with method in each subclass
‣ Facade: make client of API dependent on only a single class

where we are

‣ built a naive solver that works for small problems
‣ next time, a real SAT solver