Birefringence

Outline

- Polarized Light (Linear & Circular)
- Birefringent Materials
- Quarter-Wave Plate & Half-Wave Plate

Reading: Ch 8.5 in Kong and Shen
True / False

1. The plasma frequency $\omega_p$ is the frequency above which a material becomes a plasma.  

2. The magnitude of the $\vec{E}$-field of this wave is 1

\[ \vec{E} = (\hat{x} + \hat{y})e^{j(\omega t - k z)} \]

3. The wave above is polarized $45^\circ$ with respect to the x-axis.
**Microscopic Lorentz Oscillator Model**

\[
\epsilon = \epsilon_o \left(1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2 + j \omega \gamma}\right)
\]

\[
\tilde{\epsilon} = \epsilon_r - j \epsilon_i
\]

\[
\omega_p^2 = \frac{Nq^2}{\epsilon_o m}
\]

\[
\omega_o^2 = \frac{k_{spring}}{m}
\]
Sinusoidal Uniform Plane Waves

\[ E_y = A_1 \cos(\omega t - kz) \]

\[ E_x = A_2 \cos(\omega t - kz) \]
45° Polarization

The complex amplitude, $E_0$, is the same for both components. Therefore $E_x$ and $E_y$ are always in phase.

Where is the magnetic field?
Superposition of Sinusoidal Uniform Plane Waves

\[
\mathbf{E} = A (\cos(\omega t - kz) \mathbf{\hat{y}} + \cos(\omega t - kz) \mathbf{\hat{x}})
\]

Can it only be at 45°?
**Arbitrary-Angle Linear Polarization**

\[
E_x(z, t) = \hat{x} \text{Re}\{\tilde{E}_o \cos(\phi) \exp[j(\omega t - kz)]\}
\]

\[
E_y(z, t) = \hat{y} \text{Re}\{\tilde{E}_o \sin(\phi) \exp[j(\omega t - kz)]\}
\]

Here, the $y$-component is in phase with the $x$-component, but has different magnitude.
**Arbitrary-Angle Linear Polarization**

\[ E_x(z, t) = \hat{x} \, \text{Re}\{\tilde{E}_0 \cos(\phi) \exp[j(\omega t - kz)]\} \]

\[ E_y(z, t) = \hat{y} \, \text{Re}\{\tilde{E}_0 \sin(\phi) \exp[j(\omega t - kz)]\} \]

Specifically:
- 0° linear (x) polarization: \( E_y / E_x = 0 \)
- 90° linear (y) polarization: \( E_y / E_x = \infty \)
- 45° linear polarization: \( E_y / E_x = 1 \)
- Arbitrary linear polarization: \( E_y / E_x = \text{constant} \)

\[ \frac{E_y(z, t)}{E_x(z, t)} = \frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) \]
**Circular (or Helical) Polarization**

\[ E_x(z, t) = \hat{x} \tilde{E}_o \sin(\omega t - kz) \]
\[ E_y(z, t) = \hat{y} \tilde{E}_o \cos(\omega t - kz) \]

... or, more generally,

\[ E_x(z, t) = \hat{x} \text{Re}\{ -j \tilde{E}_o e^{j(\omega t - kz)} \} \]
\[ E_y(z, t) = \hat{y} \text{Re}\{ j \tilde{E}_o e^{j(\omega t - kz)} \} \]

The complex amplitude of the x-component is \(-j\) times the complex amplitude of the y-component.

\( E_x \) and \( E_y \) are always \(90^\circ\) out of phase

The resulting E-field rotates **counterclockwise** around the propagation-vector (looking along z-axis).

If projected on a constant z plane the E-field vector would rotate **clockwise** !!!
Right vs. Left Circular (or Helical) Polarization

\[ E_x(z, t) = -\hat{x} \tilde{E}_o \sin(\omega t - kz) \]
\[ E_y(z, t) = \hat{y} \tilde{E}_o \cos(\omega t - kz) \]

... or, more generally,

\[ E_x(z, t) = \hat{x} \Re\{+j \tilde{E}_o e^{i(\omega t - kz)}\} \]
\[ E_y(z, t) = \hat{y} \Re\{j \tilde{E}_o e^{i(\omega t - kz)}\} \]

Here, the complex amplitude of the \( x \)-component is \(+j\) times the complex amplitude of the \( y \)-component.

So the components are always \( 90^\circ \) out of phase, but in the other direction.

The resulting E-field rotates clockwise around the propagation-vector (looking along \( z \)-axis).

If projected on a constant \( z \) plane the E-field vector would rotate counterclockwise !!!
Unequal arbitrary-relative-phase components yield elliptical polarization

\[ E_x(z, t) = \hat{x} E_{ox} \cos(\omega t - k z) \]
\[ E_y(z, t) = \hat{y} E_{oy} \cos(\omega t - k z - \theta) \]

where \( E_{ox} \neq E_{oy} \)

... or, more generally,
\[ E_x(z, t) = \hat{x} \text{Re}\{E_{ox}\,e^{j(\omega t - k z)}\} \]
\[ E_y(z, t) = \hat{y} \text{Re}\{E_{oy}\,e^{j(\omega t - k z - \theta)}\} \]

... where \( \tilde{E}_{ox} \) and \( \tilde{E}_{oy} \) are arbitrary complex amplitudes

The resulting E-field can rotate clockwise or counter-clockwise around the k-vector (looking along k).
Sinusoidal Uniform Plane Waves

\[ \overline{E} = A \left( \sin(\omega t - kz) \hat{y} \pm \cos(\omega t - kz) \hat{x} \right) \]

\text{Left} \quad \text{IEEE Definitions:} \quad \text{Right}

\[ \overline{E}_L = E_o \left( \hat{x} + \hat{y}e^{+j\pi/2} \right) e^{-jkz} \quad \overline{E}_R = E_o \left( \hat{x} + \hat{y}e^{-j\pi/2} \right) e^{-jkz} \]
A linearly polarized wave can be represented as a sum of two circularly polarized waves.
A linearly polarized wave can be represented as a sum of two circularly polarized waves

\[ E_x(z, t) = \hat{x}\tilde{E}_o\sin(\omega t - k z) \]
\[ E_y(z, t) = \hat{y}\tilde{E}_o\cos(\omega t - k z) \]

\[ E_x(z, t) = -\hat{x}\tilde{E}_o\sin(\omega t - k z) \]
\[ E_y(z, t) = \hat{y}\tilde{E}_o\cos(\omega t - k z) \]
Polarizers for Linear and Circular Polarizations

CASE 1:
Linearly polarized light with magnitude $E_o$ oriented 45° with respect to the x-axis.

CASE 2:
Circularly polarized light with magnitude $E_o$.

What is the average power at the input and output?
Today’s Culture Moment

Image is in the public domain.

3D vision
The molecular "spring constant" can be different for different directions.

If $\omega_{ox} = \omega_{oz}$, then the material has a single optics axis and is called **uniaxial** crystal.
**Microscopic Lorentz Oscillator Model**

\[
P(\omega) = \frac{\epsilon_o \omega_p^2}{\omega_o^2 - \omega^2 + j \omega \gamma} E(\omega) = (\bar{\epsilon}(\omega) - 1) \epsilon_o E(\omega)
\]

In the transparent regime
\[\epsilon_{yr} \gg \epsilon_{yi} \ldots\]

\[
\eta_y = \sqrt{\frac{\epsilon_{yr}}{\epsilon_o}}
\]

\[
\eta_x = \sqrt{\frac{\epsilon_{yr}}{\epsilon_o}}
\]
Uniaxial crystals have one refractive index for light polarized along the optic axis ($n_e$) and another for light polarized in either of the two directions perpendicular to it ($n_o$).

Light polarized along the optic axis is called the extraordinary ray, and light polarized perpendicular to it is called the ordinary ray.

These polarization directions are the crystal principal axes.

Ordinary...

$$n_x = n_z = n_o$$

Extraordinary...

$$n_y = n_e$$
**Birefringent Materials**

All transparent crystals with non-cubic lattice structure are birefringent.

<table>
<thead>
<tr>
<th>Crystal</th>
<th>$\lambda = 583,nm$</th>
<th>$n_o$</th>
<th>$n_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tourmaline</td>
<td></td>
<td>1.669</td>
<td>1.638</td>
</tr>
<tr>
<td>Calcite</td>
<td></td>
<td>1.6584</td>
<td>1.4864</td>
</tr>
<tr>
<td>Quartz</td>
<td></td>
<td>1.5443</td>
<td>1.5534</td>
</tr>
<tr>
<td>Sodium nitrate</td>
<td></td>
<td>1.5854</td>
<td>1.3369</td>
</tr>
<tr>
<td>Ice</td>
<td></td>
<td>1.309</td>
<td>1.313</td>
</tr>
<tr>
<td>Rutile (TiO$_2$)</td>
<td></td>
<td>2.616</td>
<td>2.903</td>
</tr>
</tbody>
</table>

Polarization Conversion
Linear to Circular

\[ \vec{E} = E_o (\hat{x} + \hat{y}) e^{-jkz} \]

\[ \vec{E} = E_o \left( \hat{x}e^{-jk_ed} + \hat{y}e^{-jk_odo} \right) e^{-jkz} \]

inside

\[ \vec{E} = E_o \left( \hat{x}e^{-jk_ez} + \hat{y}e^{-jk_oz} \right) \]

Polarization of output wave is determined by...

\[ \frac{E_y}{E_x} = \frac{e^{-jk_ed}}{e^{-jk_odo}} = e^{-j(k_o-k_e)d} \]
**Quarter-Wave Plate**

**Example:**

If we are to make quarter-wave plate using calcite \((n_o = 1.6584, n_e = 1.4864)\), for incident light wavelength of \(\lambda = 590\) nm, how thick would the plate be?

\[
d_{\text{calcite QWP}} = \frac{590\text{nm}/4}{n_o - n_e} = 858\text{ nm}
\]
**Half-Wave Plate**

The phase difference between the waves linearly polarized parallel and perpendicular to the optic axis is a half cycle

\[
\vec{E} = E_0 (\hat{x} - \hat{y}) e^{-jk(z-d)}
\]

\[
\vec{E} = E_0 (\hat{x} + \hat{y}) e^{-jkz}
\]

LINEAR IN $\rightarrow$ LINEAR OUT
Key Takeaways

**EM Waves** can be linearly, circularly, or elliptically polarized.

A circularly polarized wave can be represented as a sum of two linearly polarized waves having $\pi/2$ phase shift.

A linearly polarized wave can be represented as a sum of two circularly polarized waves.

In the general case, waves are elliptically polarized.

Waveplates can be made from birefringent materials:

Quarter wave plate:  \[ \frac{\lambda}{4} = (n_o - n_e)d \] (gives $\pi/2$ phase shift)

Half wave plate:  \[ \frac{\lambda}{2} = (n_o - n_e)d \] (gives $\pi$ phase shift)