Tunneling

Outline

- Review: Barrier Reflection
- Barrier Penetration (Tunneling)
- Flash Memory
**A Simple Potential Step**

CASE I: \( E_o > V \)

In Region 1:

\[
E_o \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}
\]

\[ \implies k_1^2 = \frac{2mE_o}{\hbar^2} \]

In Region 2:

\[
(E_o - V) \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}
\]

\[ \implies k_2^2 = \frac{2m(E_o - V)}{\hbar^2} \]
**A Simple Potential Step**

**CASE I :** $E_0 > V$

Region 1

Region 2

$\psi_1 = Ae^{-jk_1x} + Be^{jk_1x}$

$\psi_2 = Ce^{-jk_2x}$

$\psi$ is continuous:

$\psi_1(0) = \psi_2(0)$

$A + B = C$

$\frac{\partial \psi}{\partial x}$ is continuous:

$\frac{\partial}{\partial x} \psi(0) = \frac{\partial}{\partial x} \psi_2(0)$

$A - B = \frac{k_2}{k_1} C$
A Simple Potential Step

CASE I: \( E_o > V \)

\[
\begin{align*}
\psi_A &= Ae^{-jk_1x} \\
\psi_B &= Be^{-jk_1x} \\
\psi_C &= Ce^{-jk_1x}
\end{align*}
\]

\[
\begin{align*}
\frac{B}{A} &= \frac{1 - k_2/k_1}{1 + k_2/k_1} \\
&= \frac{k_1 - k_2}{k_1 + k_2} \\
\frac{C}{A} &= \frac{2}{1 + k_2/k_1} \\
&= \frac{2k_1}{k_1 + k_2}
\end{align*}
\]

\[
\begin{align*}
A + B &= C \\
A - B &= \frac{k_2}{k_1} C
\end{align*}
\]
Example from: http://phet.colorado.edu/en/get-phet/one-at-a-time
Given an electron of mass $m$

that is located in space with charge density $\rho = q|\psi(x)|^2$

and moving with momentum $<p>$ corresponding to $<v> = \hbar k/m$

... then the current density for a single electron is given by

$$J = \rho v = q|\psi|^2 \left(\frac{\hbar k}{m}\right)$$
**A Simple Potential Step**

**CASE I :** \( E_o > V \)

![Diagram](image)

\[ \psi_A = A e^{-j k_1 x} \]
\[ \psi_B = B e^{-j k_1 x} \]
\[ \psi_C = C e^{-j k_1 x} \]

Reflection = \( R = \frac{J_{\text{reflected}}}{J_{\text{incident}}} = \frac{J_B}{J_A} = \frac{|\psi_B|^2(\hbar k_1/m)}{|\psi_A|^2(\hbar k_1/m)} = |\frac{B}{A}|^2 \)

Transmission = \( T = \frac{J_{\text{transmitted}}}{J_{\text{incident}}} = \frac{J_C}{J_A} = \frac{|\psi_C|^2(\hbar k_2/m)}{|\psi_A|^2(\hbar k_1/m)} = |\frac{C}{A}|^2 \frac{k_2}{k_1} \)

\[ \frac{B}{A} = \frac{1 - k_2/k_1}{1 + k_2/k_1} \]
\[ \frac{C}{A} = \frac{2}{1 + k_2/k_1} \]
A Simple Potential Step

CASE I: $E_o > V$

\[ \psi_A = A e^{-j k_1 x} \quad \rightarrow \quad \psi_C = C e^{-j k_1 x} \]

\[ \psi_B = B e^{-j k_1 x} \]

\[ \text{Reflection} = R = \left| \frac{B}{A} \right|^2 = \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2 \]

\[ \text{Transmission} = T = 1 - R \]

\[ \frac{k_2}{k_1} = \sqrt{1 - \frac{V}{E_o}} \]

$E = 0$  
Region 1 Region 2

$V$

$x = 0$

$E = 0$

$E_o = V$  
$E_o = \infty$

$T + R = 1$

Reflection $R$  
Transmission $T$
**A Simple Potential Step**

CASE II: $E_o < V$

\[ \psi_A = A e^{-j k_1 x} \]
\[ \psi_B = B e^{-j k_1 x} \]
\[ \psi_C = C e^{-\kappa x} \]

In Region 1:
\[ E_o \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \]
\[ \Rightarrow \quad k_1^2 = \frac{2mE_o}{\hbar^2} \]

In Region 2:
\[ (E_o - V) \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \]
\[ \Rightarrow \quad \kappa^2 = \frac{2m(E_o - V)}{\hbar^2} \]
**A Simple Potential Step**

CASE II: $E_o < V$

\[
\begin{align*}
\psi_A &= Ae^{-jk_1x} \\
\psi_B &= Be^{-jk_1x} \\
\psi_C &= Ce^{-\kappa x}
\end{align*}
\]

\[E = 0 \quad \text{Region 1} \]
\[E = E_o \quad \text{Region 2} \]

\[
\begin{align*}
\psi_1 &= Ae^{-jk_1x} + Be^{jk_1x} \\
\psi_2 &= Ce^{-\kappa x}
\end{align*}
\]

\[
\begin{align*}
\psi \text{ is continuous:} & \quad \psi_1(0) = \psi_2(0) \quad \Rightarrow \quad A + B = C \\
\frac{\partial \psi}{\partial x} \text{ is continuous:} & \quad \frac{\partial}{\partial x} \psi(0) = \frac{\partial}{\partial x} \psi_2(0) \quad \Rightarrow \quad A - B = -j\frac{\kappa}{k_1}C
\end{align*}
\]
**A Simple Potential Step**

CASE II : \( E_0 < V \)

\[
\psi_A = A e^{-j k_1 x} \quad \psi_C = C e^{-\kappa x}
\]

\[
\psi_B = B e^{-j k_1 x}
\]

Region 1: \( E = 0 \)

Region 2: \( E = E_0 \)

\[
\frac{B}{A} = \frac{1 + j \kappa / k_1}{1 - j \kappa / k_1} \quad \frac{C}{A} = \frac{2}{1 - j \kappa / k_1}
\]

\[
R = \left| \frac{B}{A} \right|^2 = 1 \quad T = 0
\]

Total reflection \( \rightarrow \) Transmission must be zero
**Quantum Tunneling** *Through a Thin Potential Barrier*

**Total Reflection at Boundary**

\[ R = 1 \]

\[ T = 0 \]

**Frustrated Total Reflection (Tunneling)**

\[ 2a = L \]

\[ T \neq 0 \]
A Rectangular Potential Step

CASE II: $E_o < V$

In Regions 1 and 3:

$$E_o \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \quad \Rightarrow \quad k_1^2 = \frac{2mE_o}{\hbar^2}$$

In Region 2:

$$(E_o - V) \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \quad \Rightarrow \quad \kappa^2 = \frac{2m(V - E_o)}{\hbar^2}$$

for $E_o < V$:

$$T = \left| \frac{F}{A} \right|^2 = \frac{1}{1 + \frac{1}{4} \frac{V^2}{E_o(V-E_o)} \sinh^2(2\kappa a)}$$
A Rectangular Potential Step

Real part of $\Psi$ for $E_o < V$, shows hyperbolic (exponential) decay in the barrier domain and decrease in amplitude of the transmitted wave.

$$E = 0$$

for $E_o < V$:

$$T = \left| \frac{F}{A} \right|^2 = \frac{1}{1 + \frac{1}{4} \frac{V^2}{E_o(V-E_o)} \sinh^2(2\kappa a)}$$

$$\sinh^2(2\kappa a) = \left[ e^{2\kappa a} - e^{-2\kappa a} \right]^2 \approx e^{-4\kappa a}$$

$$T = \left| \frac{F}{A} \right|^2 \approx \frac{1}{1 + \frac{1}{4} \frac{V^2}{E_o(V-E_o)}} e^{-4\kappa a}$$

Tunneling Applet: http://www.colorado.edu/physics/phet/dev/quantum-tunneling/1.07.00/
Flash Memory

Electrons tunnel preferentially when a voltage is applied

Image is in the public domain
**MOSFET: Transistor in a Nutshell**

Conduction electron flow

Tunneling causes thin insulating layers to become leaky!

Image is in the public domain

Image courtesy of J. Hoyt Group, EECS, MIT. Photo by L. Gomez
Reading Flash Memory

To obtain the same channel charge, the programmed gate needs a higher control-gate voltage than the unprogrammed gate.

How do we WRITE Flash Memory?
Example: Barrier Tunneling

- Let’s consider a tunneling problem:

An electron with a total energy of $E_o = 6 \text{ eV}$ approaches a potential barrier with a height of $V_0 = 12 \text{ eV}$. If the width of the barrier is $L = 0.18 \text{ nm}$, what is the probability that the electron will tunnel through the barrier?

$$T = \left| \frac{F}{A} \right|^2 \approx \frac{16E_o(V - E_o)}{V^2} e^{-2\kappa L}$$

$$\kappa = \sqrt{\frac{2m_e}{\hbar^2}}(V - E_o) = 2\pi \sqrt{\frac{2m_e}{\hbar^2}}(V - E_o) = 2\pi \sqrt{\frac{6\text{eV}}{1.505\text{eV-nm}^2}} \approx 12.6 \text{ nm}^{-1}$$

$$T = 4e^{-2(12.6 \text{ nm}^{-1})(0.18 \text{ nm})} = 4(0.011) = 0.044 \%$$

**Question:** What will $T$ be if we double the width of the gap?
Multiple Choice Questions

Consider a particle tunneling through a barrier:

1. Which of the following will increase the likelihood of tunneling?
   a. decrease the height of the barrier
   b. decrease the width of the barrier
   c. decrease the mass of the particle

2. What is the energy of the particles that have successfully “escaped”?
   a. $< \text{ initial energy}$
   b. $= \text{ initial energy}$
   c. $> \text{ initial energy}$

Although the amplitude of the wave is smaller after the barrier, no energy is lost in the tunneling process.
Application of Tunneling: Scanning Tunneling Microscopy (STM)

Due to the quantum effect of “barrier penetration,” the electron density of a material extends beyond its surface:

One can exploit this to measure the electron density on a material’s surface:

Sodium atoms on metal:

 STM images

Single walled carbon nanotube:

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Reflection of EM Waves and QM Waves

\[ P = \hbar \omega \times \frac{\text{photons}}{\text{s cm}^2} \]

\[ P = \frac{|E|^2}{\eta} \]

\[ R = \frac{P_{\text{reflected}}}{P_{\text{incident}}} = \left| \frac{E_o^r}{E_o^i} \right|^2 \]

Then for optical material when \( \mu = \mu_0 \):

\[ R = \frac{B}{A} = \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2 = \left| n_1 + n_2 \right|^2 \left| \frac{n_1 + n_2}{n_1 + n_2} \right| = \text{probability of a particular photon being reflected} \]

\[ J = q \times \frac{\text{electrons}}{\text{s cm}^2} \]

\[ J = \rho \nu = q |\psi|^2 (\hbar k/m) \]

\[ R = \frac{J_{\text{reflected}}}{J_{\text{incident}}} = \left| \frac{\psi_B}{\psi_A} \right|^2 \]

\[ R = \frac{B}{A} = \left| \frac{k_1 - k_2}{k_1 + k_2} \right|^2 = \left| \frac{n_1 + n_2}{n_1 + n_2} \right| = \text{probability of a particular electron being reflected} \]