Problem 6.1
An electric field is present within a plasma of dielectric permittivity $\varepsilon$ with conduction constituent relation

$$\frac{\partial \mathbf{J}_f}{\partial t} = \omega_p^2 \varepsilon \mathbf{E}, \text{ where } \omega_p^2 = \frac{q^2 n}{m \varepsilon}$$

with $q, n$ and $m$ being the charge, number density (number per unit volume) and mass of each charge carrier.

(a) Poynting’s theorem is

$$\nabla \cdot \mathbf{S} + \frac{\partial w_{EM}}{\partial t} = -\mathbf{E} \cdot \mathbf{J}_f$$

For the plasma medium, $\mathbf{E} \cdot \mathbf{J}_f$, can be written as

$$\mathbf{E} \cdot \mathbf{J}_f = \frac{\partial w_k}{\partial t}.$$

What is $w_k$?

(b) What is the velocity $v$ of the charge carriers in terms of the current density $\mathbf{J}_f$ and parameters $q, n$ and $m$ defined above?

(c) Write $w_k$ of part (a) in terms of $v$, $q$, $n$, and $m$. What kind of energy density is $w_k$?

(d) Assuming that all fields vary sinusoidally with time as:

$$\mathbf{E}(r, t) = \text{Re} \left[ \hat{\mathbf{E}}(r) e^{j\omega t} \right]$$

write Maxwell’s equations in complex amplitude form with the plasma constitutive law.

(e) Reduce the complex Poynting theorem from the usual form

$$\nabla \left[ \frac{1}{2} \mathbf{E}(r) \times \mathbf{H}^*(r) \right] + 2j \omega < w_{EM} >= -\frac{1}{2} \mathbf{E} \cdot \mathbf{J}_f^*$$
to
\[ \nabla \left[ \frac{1}{2} \hat{\mathbf{E}}(r) \times \hat{\mathbf{H}}^*(r) \right] + 2j\omega (\langle w_{EM} \rangle + \langle w_k \rangle) = 0 \]

What are \( \langle w_{EM} \rangle \) and \( \langle w_k \rangle \)?

(f) Show that
\[ \langle w_{EM} \rangle + \langle w_k \rangle = \frac{1}{4} \mu \mathbf{H}^2 - \frac{1}{4} \varepsilon(\omega) \mathbf{E}^2 \]

What is \( \varepsilon(\omega) \) and compare to the results from Problem 5.3b?

Problem 6.2

A TEM wave \((E_x, H_y)\) propagates in a medium whose dielectric permittivity and magnetic permeability are functions of \(z, \varepsilon(z)\) and \(\mu(z)\).

(a) Write down Maxwell’s equations and obtain a single partial differential equation in \(H_y\).

(b) Consider the idealized case where \(\varepsilon(z) = \varepsilon_a e^{az}\) and \(\mu(z) = \mu_a e^{-az}\). Show that the equation of (a) for \(H_y\) reduces to a linear partial differential equation with constant coefficients of the form
\[ \frac{\partial^2 H_y}{\partial z^2} - \beta \frac{\partial H_y}{\partial z} - \gamma \frac{\partial^2 H_y}{\partial t^2} = 0 \]

What are \(\beta\) and \(\gamma\)?

(c) Infinite magnetic permeability regions with zero magnetic field extend for \(z<0\) and \(z>d\). A current sheet \(\text{Re}\left[\tilde{J}_y K_0 e^{j\omega t}\right]\) is placed at \(z = 0\). Take the magnetic field of the form
\[ \mathbf{H} = \text{Re}[\tilde{J}_y \hat{H}_y e^{j(\omega t - kx)}] \]

and find values of \(\kappa\) that satisfy the governing equation in (b) for \(0<z<d\).

(d) What are the boundary conditions on \(\mathbf{H}\)?

(e) Superpose the solutions found in (c) and find \(\mathbf{H}\) that satisfies the boundary conditions of (d).

(f) What is the electric field for \(0<z<d\)?
Problem 6.3

A sheet of surface charge with charge density \( \sigma_f = \text{Re}[\hat{\sigma}_0 e^{j(\omega t - k_y y)}] \) is placed in free space \((\varepsilon_0, \mu_0)\) at \( z = 0 \).

The complex magnetic field in each region is of the form

\[
\vec{H} = \begin{cases} 
\hat{H}_1 e^{-j(k_y y + k_z z)} & z > 0 \\
\hat{H}_2 e^{j(k_y y - k_z z)} & z < 0 
\end{cases}
\]

(a) What is \( k_z \)?

(b) What is the complex electric field for \( z < 0 \) and \( z > 0 \) in terms of \( \hat{H}_1, \hat{H}_2, k_y, k_z \) and \( \omega \)?

(c) Using the boundary conditions at \( z = 0 \), what are \( \hat{H}_1 \) and \( \hat{H}_2 \)?

(d) For what range of the frequency will the waves for \( z < 0 \) and \( z > 0 \) be evanescent?

(e) What surface current flows on the charge sheet at \( z = 0 \)?
Problem 6.4

A TM wave is incident onto a medium with a dielectric permittivity $\epsilon_2$ from a medium with dielectric permittivity $\epsilon_1$ at the Brewster’s angle of no reflection, $\theta_B$. Both media have the same magnetic permeability $\mu_1 = \mu_2 = \mu$. The reflection coefficient for a TM wave is

$$\frac{\hat{E}_r}{\hat{E}_i} = R = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_r}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_r}$$

(a) What is the transmitted angle $\theta_t$ when $\theta_i = \theta_B$? How are $\theta_B$ and $\theta_t$ related?

(b) What is the Brewster angle of no reflection?

(c) What is the critical angle of transmission $\theta_C$ when $\mu_1 = \mu_2 = \mu$? For the critical angle to exist, what must be the relationship between $\epsilon_1$ and $\epsilon_2$?
(d) A Brewster prism will pass TM polarized light without any loss from reflections.

For the light path through the prism shown above what is the apex angle $\theta$? Evaluate for glass with $n = 1.45$.

(e) In the Brewster prism of part (d), determine the output power in terms of the incident power for TE polarized light with $n = 1.45$. The reflection coefficient for a TE wave is

$$\frac{\hat{E}_r}{\hat{E}_i} = R = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i}.$$