I. Wave equation (Lossless)

\[ \frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t} \Rightarrow \frac{\partial^2 v}{\partial t^2} = \frac{1}{2}\frac{\partial^2 v}{\partial z^2} \]

\[ c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon \mu}} \]

Solution: \( v(z,t) = V_+ \left( t - \frac{z}{c} \right) + V_- \left( t + \frac{z}{c} \right) \)

Proof: Let \( \alpha = t - \frac{z}{c} \Rightarrow \frac{\partial v}{\partial t} = \alpha, \frac{\partial v}{\partial z} = -\frac{1}{c} \)

Superposition: \( v_+(z,t) = \frac{d v_+}{d \alpha} = \frac{d v_+}{d \alpha} \)

\[ \frac{\partial^2 v_+}{\partial z^2} = \frac{c^2 \partial^2 v}{\partial z^2} = c^2 \frac{d^2 v_+}{d \alpha^2} = \frac{d^2 v_+}{d \alpha^2} \]

II. Solution for current \( i(z,t) \)

\[ \frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t} \Rightarrow \frac{\partial^2 i}{\partial t^2} = \frac{c^2 \partial^2 i}{\partial z^2} \]
Solution: $i(z, t) = I_+ (t - \frac{z}{c}) + I_- (t + \frac{z}{c})$
$v(z, t) = V_+ (t - \frac{z}{c}) + V_- (t + \frac{z}{c})$

$+z$ solution: $\alpha = t - \frac{z}{c}, \frac{\partial \alpha}{\partial t} = 1, \frac{\partial \alpha}{\partial z} = -\frac{1}{c}$

\[
\frac{\partial v_+}{\partial z} = -L \frac{\partial i_+}{\partial t} \Rightarrow \frac{dv_+}{\partial \alpha} = \frac{1}{c} \frac{dv_+}{\partial \alpha} = -L \frac{di_+}{\partial \alpha} \frac{\partial \alpha}{\partial t}
\]

\[
\frac{dv_+}{\partial \alpha} = \frac{Lc}{\sqrt{LC}} \frac{di_+}{\partial \alpha} = \sqrt{\frac{L}{C}} \frac{di_+}{\partial \alpha} = Z_0 \frac{di_+}{\partial \alpha}
\]

$v_+ = i_+ Z_0 \Rightarrow I_+ (t - \frac{z}{c}) = Y_0 V_+ (t - \frac{z}{c})$

$Y_0 = \sqrt{\frac{C}{L}} = \frac{1}{Z_0}$

$-z$ solution: $\beta = t + \frac{z}{c}, \frac{\partial \beta}{\partial t} = 1, \frac{\partial \beta}{\partial z} = \frac{1}{c}$

\[
\frac{\partial v_-}{\partial z} = -L \frac{\partial i_-}{\partial t} \Rightarrow \frac{dv_-}{\partial \beta} = \frac{1}{c} \frac{dv_-}{\partial \beta} = -L \frac{di_-}{\partial \beta} \frac{\partial \beta}{\partial t}
\]

\[
\frac{dv_-}{\partial \beta} = -Lc \frac{di_-}{\partial \beta} = - \sqrt{\frac{L}{LC}} \frac{di_-}{\partial \beta} = -\sqrt{\frac{L}{C}} \frac{di_-}{\partial \beta} = -Z_0 \frac{di_-}{\partial \beta}
\]

$v_- = -i_- Z_0 \Rightarrow I_+ \left(t + \frac{z}{c}\right) = -Y_0 V_- \left(t + \frac{z}{c}\right)$

$v(z, t) = V_+ \left(t - \frac{z}{c}\right) + V_- \left(t + \frac{z}{c}\right)$

$i(z, t) = Y_0 \left[V_+ \left(t - \frac{z}{c}\right) - V_- \left(t + \frac{z}{c}\right)\right]$
With $V_-(t + \frac{z}{c}) = 0 \Rightarrow v(z, t) = V_+ \left(t - \frac{z}{c}\right) \Rightarrow \frac{v(z, t)}{i(z, t)} = Z_0$

$i(z, t) = Y_0 V_+ \left(t - \frac{z}{c}\right)$

2. Traveling Wave Solution

$v(z = 0, t) = V(t) = V_+(t)$
$v(z = 0, t) = \frac{Z_0}{Z_0 + R_S} V(t) = V_+(t)$
$i(z = 0, t) = Y_0 V_+(t) = \frac{V(t)}{R_S + Z_0}$
$v(z, t) = \frac{Z_0}{Z_0 + R_S} V \left(t - \frac{z}{c}\right)$
$i(z, t) = \frac{1}{R_S + Z_0} V \left(t - \frac{z}{c}\right)$

![Figure 8-6](image)

(a) A semi-infinite transmission line excited by a voltage source at $z = 0$. (b) To the source, the transmission line looks like a resistor $Z_0$ equal to the characteristic impedance. (c) The spatial distribution of the voltage $v(z, t)$ at various times for a staircase pulse of $V(t)$. (d) If the voltage source is applied to the transmission line through a series resistance $R_s$, the voltage across the line at $z = 0$ is given by the voltage divider relation.

B. Reflections from Resistive Terminations

1. Reflection Coefficient

At $z = l$:

$$v(z, t) = V_+ \left(t - \frac{l}{c}\right) + V_- \left(t + \frac{l}{c}\right)$$

$$= i(l, t)R_L$$

$$= Y_0R_L \left[V_+ \left(t - \frac{l}{c}\right) - V_- \left(t + \frac{l}{c}\right)\right]$$

$$\Gamma_L = \frac{V_- (t + \frac{l}{c})}{V_+ (t - \frac{l}{c})} = \frac{R_L - Z_0}{R_L + Z_0}$$

Special cases:

a. $R_L = Z_0 \Rightarrow \Gamma_L = 0$ (matched line)

b. $R_L = 0 \Rightarrow \Gamma_L = -1$ (short circuited line)

If $R_L < Z_0, \Gamma_L < 0$

c. $R_L = \infty \Rightarrow \Gamma_L = +1$ (open circuited line)

If $R_L > Z_0, \Gamma_L > 0$

2. Step Voltage

At $z = 0$:

$$v(z = 0, t) + i(0, t)R_S = V_0$$

$$V_+(z = 0, t) + V_-(z = 0, t) + Y_0R_S [V_+(z = 0, t) - V_-(z = 0, t)] = V_0$$

$$V_+(z = 0, t) = \Gamma_S V_-(z = 0, t) + \frac{Z_0V_0}{Z_0 + R_S}, \Gamma_S = \frac{R_S - Z_0}{R_S + Z_0}$$

a. Matched Line: $R_L = Z_0, \Gamma_L = 0; R_S = Z_0, \Gamma_S = 0$

$$\Gamma_L = 0 \Rightarrow V_- (t + \frac{z}{c}) = 0, V_+ (z = 0, t) = \frac{V_0}{2}, \text{ in steady state after time } T = \frac{l}{c}$$

b. Short circuited line: $R_L = 0, \Gamma_L = -1, R_S = Z_0, \Gamma_S = 0$

$$\Gamma_L = -1 \Rightarrow V_+ = -V_- \text{. When } V_+ (t - \frac{z}{c}) \text{ and } V_- (t + \frac{z}{c}) \text{ overlap in space, } v(z, t) = 0. \text{ For } t \geq 2T = \frac{2l}{c}, v(z, t) = 0, i(z, t) = \frac{V_0}{Z_0}.$$

c. Open circuited line: $R_L = \infty, \Gamma_L = +1, R_S = Z_0, \Gamma_S = 0$

$$\Gamma_L = +1 \Rightarrow V_+ = +V_- \text{. For } t \geq 2T = \frac{2l}{c}, v(z, t) = V_0, i(z, t) = 0$$
Figure 8-8  (a) A dc voltage $V_0$ is switched onto a resistively loaded transmission line through a source resistance $R_S$. (b) The equivalent circuits at $z = 0$ and $z = l$ allow us to calculate the reflected voltage wave amplitudes in terms of the incident waves.


Figure 8-9  (a) A dc voltage is switched onto a transmission line with load resistance $R_L$ through a source resistance $R_S$ matched to the line. (b) Regardless of the load resistance, half the source voltage propagates down the line towards the load. If the load is also matched to the line ($R_L = Z_0$), there are no reflections and the steady state of $V(z, t = T) = V_0/2$, $i(z, t = T) = Y_0 V_0/2$ is reached for $t \geq T$. (c) If the line is short circuited ($R_L = 0$), then $\Gamma_L = -1$ so that the $V_+$ and $V_-$ waves cancel for the voltage but add for the current wherever they overlap in space. Since the source end is matched, no further reflections arise at $z = 0$ so that the steady state is reached for $t \geq 2T$. (d) If the line is open circuited ($R_L = \infty$) so that $\Gamma_L = +1$, the $V_+$ and $V_-$ waves add for the voltage but cancel for the current.

3. Approach to the DC Steady State (neither end matched)

\[ z = 0 : V_+(t) = \Gamma_0 V_0 + \Gamma_S V_-(t), \Gamma_0 = \frac{Z_0}{R_S + Z_0}, \Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} \]

\[ z = l : V_-(t + \frac{l}{c}) = \Gamma_L V_+(t - \frac{l}{c}), \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \]

at \( z = l \):

\[ V_{+n} = \Gamma_0 V_0 + \Gamma_S V_{-(n-1)} \]

\[ V_{-(n-1)} = \Gamma_L V_{+(n-1)} \]

\[ V_{+n} = \Gamma_0 V_0 + \Gamma_S \Gamma_L V_{+(n-1)} \Rightarrow V_{+n} - \Gamma_S \Gamma_L V_{+(n-1)} = \Gamma_0 V_0 \]

Particular Solution:

\[ V_{+n} = \text{constant} \]

\[ \text{constant} \left( 1 - \Gamma_S \Gamma_L \right) = \Gamma_0 V_0 \]

\[ \text{constant} = \frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L} \]

Homogeneous Solution:

\[ V_{+n} - \Gamma_S \Gamma_L V_{+(n-1)} = 0 \]

Try a solution of the form: \( V_{+n} = A\lambda^n \)

\[ A \left( \lambda^n - \Gamma_S \Gamma_L \lambda^{n-1} \right) = 0 \Rightarrow \lambda = \Gamma_S \Gamma_L \]

\[ V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L} + A (\Gamma_S \Gamma_L)^n \]
Initial Condition:

\[ V_{+1} = \Gamma_0 V_0 = \frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L} + A(\Gamma_S \Gamma_L) \Rightarrow A = -\frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L} \]

\[ V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_S \Gamma_L} [1 - (\Gamma_S \Gamma_L)^n] \]

\[ V_{-(n-1)} = \Gamma_L V_{+(n-1)} \Rightarrow V_{-n} = \Gamma_L V_{+n} \]

\[ V_n = V_{+n} + V_{-n} = V_{+n}(1 + \Gamma_L) = \frac{V_0(1 + \Gamma_L) \Gamma_0 V_0}{1 - \Gamma_S \Gamma_L} [1 - (\Gamma_S \Gamma_L)^n] \]

\[ \lim_{n \to \infty} V_n = \frac{R_L}{R_L + R_S} V_0 \]

\[
\begin{array}{lllllll}
\text{a.} & \text{Special Case: } & R_S = 0, & R_L = 3Z_0. \\
\Gamma_S = -1, & \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{1}{3} = \frac{1}{2} \Rightarrow & \Gamma_S \Gamma_L = -\frac{1}{2}
\end{array}
\]

\[
z = t \quad V_n = V_0 \left[ 1 - \left( -\frac{1}{2} \right)^n \right] \quad I_n = \frac{V_0}{3Z_0} \left[ 1 - \left( -\frac{1}{2} \right)^n \right]
\]

\[
\begin{array}{lllllll}
\text{b.} & \text{Special Case: } & R_S = 0, & R_L = \frac{1}{3}Z_0
\end{array}
\]

\[
\Gamma_S = -1, \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = -\frac{\frac{1}{2}}{\frac{3}{4}} = -\frac{1}{2} \Rightarrow \Gamma_S \Gamma_L = +\frac{1}{2}
\]

\[
z = l \quad \quad V_n = V_0 \left[ 1 - \left( \frac{1}{2} \right)^n \right]
\]

c. Special Case: \( R_S = 0, R_L = \infty \) (open circuit)

\[
\Gamma_S \Gamma_L = -1
\]

\[
V_n = \frac{R_L}{R_S + R_L} V_0 \left[ 1 - (\Gamma_S \Gamma_L)^n \right] = V_0 (1 - (-1)^n)
\]

\[
= \begin{cases} 
0 & \text{n even} \\
2V_0 & \text{n odd}
\end{cases}
\]

d. Special Case: \( R_S = 0, R_L = 0 \) (short circuit)

\[
\Gamma_S \Gamma_L = +1
\]

\[
I_n = \frac{V_0}{R_L + R_S} \left[ 1 - (\Gamma_S \Gamma_L)^n \right] \quad \text{Indeterminate}
\]

\[
\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = \frac{\frac{R_S}{Z_0} - 1}{\frac{R_S}{Z_0} + 1} \approx - \left( 1 - \frac{R_S}{Z_0} \right)^2 \approx - \left( 1 - 2 \frac{R_S}{Z_0} \right)
\]

\[
\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \approx - \left( 1 - \frac{2R_L}{Z_0} \right)
\]

\[
I_N = \frac{V_0}{R_L + R_S} \left[ 1 - \left( 1 - \frac{2R_L}{Z_0} \right) \left( 1 - \frac{2R_S}{Z_0} \right) \right]^n
\]

\[
\approx \frac{V_0}{R_L + R_S} \left[ 1 - \left( 1 - \frac{2(R_L + R_S)}{Z_0} \right) \right]^n
\]

\[
\approx \frac{V_0}{R_L + R_S} \left[ 1 - 1 + \frac{2n(R_L + R_S)}{Z_0} \right]
\]

\[
\approx \frac{V_0 \cdot 2n}{Z_0}
\]

This approximates an inductor: \( V_0 = (LI) \frac{di}{dt} \Rightarrow i = \frac{V_0}{Li} t \)

e. Special Case: \( R_L = \infty \) (open circuit)

\[
\Gamma_L = 1 \Rightarrow V_n = V_0 [1 - \Gamma_S^n]
\]

This approximates the transmission line as a capacitor being charged through the resistor \( R_S \):

\[
v(t) = V_0 \left( 1 - e^{-t/\tau} \right)
\]

\[
\tau = R_S C l
\]
Figure 8-11  The (a) open circuit voltage and (b) short circuit current at the $z=1$ end of the transmission line for $R_s = 0$. No dc steady state is reached because the system is lossless. If the short circuit transmission line is modeled as an inductor in the quasi-static limit, a step voltage input results in a linearly increasing current (shown dashed). The exact transmission line response is the solid staircase waveform.


Figure 8-12 The open circuit voltage at $z=1$ for a step voltage applied at $t=0$ through a source resistance $R_s$ for various values of $T/\tau$, which is the ratio of propagation time $T = l/c$ to quasi-static charging time $\tau = R_s C l$. The dashed curve shows the exponential rise obtained by a circuit analysis assuming the open circuited transmission line is a capacitor.

C. Reflections from Arbitrary Terminations

\[ V_L(t) = V_0 [V_+(t - \frac{t}{c}) - V_-(t + \frac{t}{c})] \]

\[ i_L(t) = \frac{V_0}{Z_0} [V_+(t - \frac{t}{c}) - V_-(t + \frac{t}{c})] \]

Figure 8-13 A transmission line with an \((a)\) arbitrary load at the \(z = l\) end can be analyzed from the equivalent circuit in \((b)\). Since \(V_+\) is known, calculation of the load current or voltage yields the reflected wave \(V_-\).


1. Capacitor \(C_L\) at \(z = l\), \(R_S = Z_0\) \(\Rightarrow V_+ = \frac{V_0}{2}\)

\[
t > T \quad V_L(t) = v_c(t), I_L(t) = C_L \frac{dv_c}{dt} \]

\[
Z_0 C_L \frac{dv_c}{dt} + v_c = 2V_+ = V_0, t > T
\]

\[
v_c(t) = \frac{V_0}{2} \left[ 1 - e^{-\left( t - T \right)/\left( Z_0 C_L \right)} \right], t > T
\]

\[
T = \frac{l}{c}
\]

\[
V_- = v_c(t) - V_+
\]

\[
= -\frac{V_0}{2} + V_0 \left[ 1 - e^{-\left( t - T \right)/\left( Z_0 C_L \right)} \right]
\]

\[
= \frac{V_0}{2} - V_0 e^{-\left( t - T \right)/\left( Z_0 C_L \right)}
\]

\[
i_c = C_L \frac{dv_c}{dt} = \frac{V_0}{Z_0} e^{-\left( t - T \right)/\left( Z_0 C_L \right)}, t > T
\]

2. Inductor \(L_L\) at \(z = l\)

\[
L_L \frac{di_L}{dt} + i_L Z_0 = 2V_+ = V_0, t > T
\]

\[
i_L = \frac{V_0}{Z_0} \left[ 1 - e^{-\left( t - T \right) Z_0 / L_L} \right], t > T
\]

\[
v_L = L_L \frac{di_L}{dt} = V_0 e^{-\left( t - T \right) Z_0 / L_L}, t > T
\]
Figure 8-14  (a) A step voltage is applied to transmission lines loaded at $z = l$ with a capacitor $C_L$ or inductor $L_L$. The load voltage and current are calculated from the (b) resistive-capacitive or (c) resistive-inductive equivalent circuits at $z = l$ to yield exponential waveforms with respective time constants $\tau = Z_0 C_L$ and $\tau = L_L / Z_0$ as the solutions approach the dc steady state. The waveforms begin after the initial $V_i$ wave arrives at $z = l$ after a time $T = \lambda/c$. There are no further reflections as the source end is matched.