Spectral content via the DTFT
Demo: “Deconvolving” Output of Channel with Echo

Suppose channel is LTI with

\[ h_1[n] = \delta[n] + 0.8\delta[n-1] \]

\[ H_1(\Omega) = \sum_{m} h_1[m]e^{-j\Omega m} \]

= 1 + 0.8e^{-j\Omega} = 1 + 0.8\cos(\Omega) – j0.8\sin(\Omega)

So:

\[ |H_1(\Omega)| = [1.64 + 1.6\cos(\Omega)]^{1/2} \quad \text{EVEN function of } \Omega; \]

\[ <H_1(\Omega) = \arctan \left[ -(0.8\sin(\Omega)) / [1 + 0.8\cos(\Omega)] \right] \quad \text{ODD} . \]
A Frequency-Domain view of Deconvolution

Given $H_1(\Omega)$, what should $H_2(\Omega)$ be, to get $z[n]=x[n]$?

$H_2(\Omega)=1/H_1(\Omega)$  "Inverse filter"

$= (1/|H_1(\Omega)|). \exp\{-j<\Omega\}$

Inverse filter at receiver does very badly in the presence of noise that adds to $y[n]$: filter has high gain for noise precisely at frequencies where channel gain $|H_1(\Omega)|$ is low (and channel output is weak)!
DT Fourier Transform (DTFT) for Spectral Representation of General $x[n]$

If we can write

$$h[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} H(\Omega) e^{j\Omega n} \, d\Omega$$

then we can write

$$x[n] = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} X(\Omega) e^{j\Omega n} \, d\Omega$$

where

$$H(\Omega) = \sum_{m} h[m] e^{-j\Omega m}$$

$$X(\Omega) = \sum_{m} x[m] e^{-j\Omega m}$$

Any contiguous interval of length $2\pi$

This Fourier representation expresses $x[n]$ as a weighted combination of $e^{j\Omega n}$ for all $\Omega$ in $[-\pi, \pi]$.

$X(\Omega_o) d\Omega$ is the spectral content of $x[n]$ in the frequency interval $[\Omega_o, \Omega_o + d\Omega]$.
The spectrum of the exponential signal \((0.5)^n u[n]\) is shown over the frequency range \(\Omega = 2\pi f\) in \([-4\pi,4\pi]\), The angle has units of degrees.
$x[n]$ and $X(\Omega)$
Input/Output Behavior of LTI System in Frequency Domain

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \]

\[ H(\Omega) \]

\[ y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega) X(\Omega) e^{j\Omega n} d\Omega \]

\[ Y(\Omega) = H(\Omega) X(\Omega) \]

Compare with \( y[n] = (h * x)[n] \)

Again, convolution in time has mapped to multiplication in frequency
Magnitude and Angle

\[ Y(\Omega) = H(\Omega)X(\Omega) \]

\[ |Y(\Omega)| = |H(\Omega)| \cdot |X(\Omega)| \]

and

\[ <Y(\Omega) = <H(\Omega) + <X(\Omega) \]
Core of the Story

1. A huge class of DT and CT signals can be written --- using Fourier transforms --- as a weighted sums of sinusoids (ranging from very slow to very fast) or (equivalently, but more compactly) complex exponentials. The sums can be discrete $\sum$ or continuous $\int$ (or both).

2. LTI systems act very simply on sums of sinusoids: superposition of responses to each sinusoid, with the frequency response determining the frequency-dependent scaling of magnitude, shifting in phase.
Loudspeaker Bandpass Frequency Response

SPL versus Frequency

(Speaker Sensitivity = 85dB)

-3dB @ 56.5Hz
-3dB @ 12.5k Hz

Image by MIT OpenCourseWare.
Spectral Content of Various Sounds

- Human Voice
- Cymbal Crash
- Snare Drum
- Bass Drum
- Guitar
- Bass Guitar
- Synthesizer
- Piano

<table>
<thead>
<tr>
<th></th>
<th>13.75 Hz-27.5 Hz</th>
<th>27.5 Hz-55 Hz</th>
<th>55 Hz-110 Hz</th>
<th>110 Hz-220 Hz</th>
<th>220 Hz-440 Hz</th>
<th>440 Hz-880 Hz</th>
<th>880 Hz-1,760 Hz</th>
<th>1,760 Hz-3,520 Hz</th>
<th>3,520 Hz-7,040 Hz</th>
<th>7,040 Hz-14,080 Hz</th>
<th>14,080 Hz-28,160 Hz</th>
</tr>
</thead>
</table>

Image by MIT OpenCourseWare.
Connection between CT and DT

The continuous-time (CT) signal

\[ x(t) = \cos(\omega t) = \cos(2\pi ft) \]

sampled every \( T \) seconds, i.e., at a sampling frequency of \( f_s = 1/T \), gives rise to the discrete-time (DT) signal

\[ x[n] = x(nT) = \cos(\omega nT) = \cos(\Omega n) \]

So \( \Omega = \omega T \)

and \( \Omega = \pi \) corresponds to \( \omega = \pi/T \) or \( f = 1/(2T) = f_s/2 \)
Signal $x[n]$ that has its frequency content uniformly distributed in $[-\Omega_c, \Omega_c]$

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(\Omega) e^{j\Omega n} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} e^{j\Omega n} d\Omega$$

$$= \frac{\sin(\Omega_c n)}{\pi n}, \quad n \neq 0$$

$$= \Omega_c / \pi, \quad n = 0$$

DT “sinc” function
(extends to $\pm \infty$ in time, falls off only as $1/n$)
x[n] and X(Ω)
$X(\Omega)$ and $x[n]$

![Graphs showing magnitude of frequency response $|H(e^{j\Omega})|$ and corresponding impulse responses $h[n]$ for low-pass, high-pass, band-pass, and band-stop filters.](image)
Fast Fourier Transform (FFT) to compute samples of the DTFT for signals of finite duration

\[ X(\Omega_k) = \sum_{m=0}^{P-1} x[m]e^{-j\Omega_km}, \quad x[n] = \frac{1}{P} \sum_{k=-P/2}^{(P/2)-1} X(\Omega_k)e^{j\Omega_kn} \]

where \( \Omega_k = k(2\pi/P) \), P is some integer (preferably a power of 2) such that P is longer than the time interval [0,L-1] over which \( x[n] \) is nonzero, and k ranges from \(-P/2\) to \((P/2)-1\) (for even P).

Computing these series involves \( O(P^2) \) operations – when P gets large, the computations get very s l o w....

Happily, in 1965 Cooley and Tukey published a fast method for computing the Fourier transform (aka FFT, IFFT), rediscovering a technique known to Gauss. This method takes \( O(P \log P) \) operations.

\[ P = 1024, \quad P^2 = 1,048,576, \quad P \log P \approx 10,240 \]
Where do the $\Omega_k$ live?

e.g., for $P=6$ (even)

\[
\begin{align*}
\exp(j\Omega_0) &= \exp(j\Omega_1) \\
\exp(j\Omega_2) &= \exp(j\Omega_3) \\
\exp(j\Omega_{-2}) &= -j \\
\exp(j\Omega_{-3}) &= 1
\end{align*}
\]
Spectrum of Digital Transmissions

transmit @ 7 samples/bit

$|a_k|$ (scaled version of DTFT samples)

$x[n]$ synthesized from $a_k$
Spectrum of Digital Transmissions

transmit @ 7 samples/bit

$|a_k|$  

$x[n]$ synthesized from $a_k$
Observations on previous figure

• The waveform $x[n]$ cannot vary faster than the step change every 7 samples, so we expect the highest frequency components in the waveform to have a period around 14 samples. (This is rough and qualitative, as $x[n]$ is not sinusoidal.)

• A period of 14 corresponds to a frequency of $2\pi/14 = \pi/7$, which is $1/7$ of the way from 0 to the positive end of the frequency axis at $\pi$ (so $k$ approximately 100/7 or 14 in the figure). And that indeed is the neighborhood of where the Fourier coefficients drop off significantly in magnitude.

• There are also lower-frequency components corresponding to the fact that the 1 or 0 level may be held for several bit slots.

• And there are higher-frequency components that result from the transitions between voltage levels being sudden, not gradual.
Effect of Low-Pass Channel

|a_k| cutoff @ ±k = 25

x[n] synthesized from a_k

|a_k| cutoff @ ±k = 15

x[n] synthesized from a_k
How Low Can We Go?

$|a_k| \text{ cutoff } @ \pm k = 15$

$|a_k| \text{ cutoff } @ \pm k = 14$

$|a_k| \text{ cutoff } @ \pm k = 13$

$|a_k| \text{ cutoff } @ \pm k = 12$

$|a_k| \text{ cutoff } @ \pm k = 11$

7 samples/bit $\rightarrow$ 14 samples/period $\rightarrow$ $k = (N/14) = (196/14) = 14$
6.02 Introduction to EECS II: Digital Communication Systems
Fall 2012

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.