Recitation 5 Solutions
September 23, 2010

1. (a) See derivation in textbook pp. 84-85.
   (b) See derivation in textbook p. 86.
   (c) See derivation in textbook p. 87.

2. (a) \( X \) is a Binomial random variable with \( n = 10, p = 0.2 \). Therefore,

\[
p_X(k) = \binom{10}{k} 0.2^k 0.8^{10-k}, \quad \text{for } k = 0, \ldots, 10
\]

and \( p_X(k) = 0 \) otherwise.

(b) \( P(\text{No hits}) = p_X(0) = (0.8)^{10} = 0.1074 \)

(c) \( P(\text{More hits than misses}) = \sum_{k=6}^{10} p_X(k) = \sum_{k=6}^{10} \binom{10}{k} 0.2^k 0.8^{10-k} = 0.0064 \)

(d) Since \( X \) is a Binomial random variable,

\[
E[X] = 10 \cdot 0.2 = 2 \quad \text{var}(X) = 10 \cdot 0.2 \cdot 0.8 = 1.6
\]

(e) \( Y = 2X - 3 \), and therefore

\[
E[Y] = 2E[X] - 3 = 1 \quad \text{var}(Y) = 4\text{var}(X) = 6.4
\]

(f) \( Z = X^2 \), and therefore

\[
E[Z] = E[X^2] = (E[X])^2 + \text{var}(X) = 5.6
\]

3. (a) We expect \( E[X] \) to be higher than \( E[Y] \) since if we choose the student, we are more likely to pick a bus with more students.

(b) To solve this problem formally, we first compute the PMF of each random variable and then compute their expectations.

\[
p_X(x) = \begin{cases} 
40/148 & x = 40 \\
33/148 & x = 33 \\
25/148 & x = 25 \\
50/148 & x = 50 \\
0 & \text{otherwise}.
\end{cases}
\]
and \( E[X] = 40 \frac{40}{148} + 33 \frac{33}{148} + 25 \frac{25}{148} + 50 \frac{50}{148} = 39.28 \)

\[ p_Y(y) = \begin{cases} 
1/4 & y = 40, 33, 25, 50 \\
0 & otherwise.
\end{cases} \]

and \( E[Y] = 40 \frac{1}{4} + 33 \frac{1}{4} + 25 \frac{1}{4} + 50 \frac{1}{4} = 37 \)

Clearly, \( E[X] > E[Y] \).

4. The expected value of the gain for a single game is infinite since if \( X \) is your gain, then

\[ \sum_{k=1}^{\infty} 2^k \cdot 2^{-k} = \sum_{k=1}^{\infty} 1 = \infty \]

Thus if you are faced with the choice of playing for a given fee \( f \) or not playing at all, and your objective is to make the choice that maximizes your expected net gain, you would be willing to pay any value of \( f \). However, this is in strong disagreement with the behavior of individuals. In fact experiments have shown that most people are willing to pay only about $20 to $30 to play the game. The discrepancy is due to a presumption that the amount one is willing to pay is determined by the expected gain. However, expected gain does not take into account a person’s attitude towards risk taking.

Below are histograms showing the payout results for various numbers of simulations of this game: