Recitation 11 Solutions
October 14, 2010

1. We need to apply the version of Bayes rule for a discrete random variable conditioned on a continuous random variable:

\[ p_{X|Z}(x \mid z) = \frac{p_X(x) f_{Z|X}(z \mid x)}{f_Z(z)} = \frac{p_X(x) f_{Z|X}(z \mid x)}{\sum_{k=0}^{\infty} p_X(k) f_{Z|X}(z \mid k)} \]

Specifically,

\[ P(X = 1 \mid Z = z) = p_{X|Z}(1 \mid z) = \frac{p_X(1) f_{Z|X}(z \mid 1)}{\sum_{k=0}^{\infty} p_X(k) f_{Z|X}(z \mid k)} \]

\[ = \frac{p \frac{1}{2} \lambda e^{-\lambda |z-1|}}{(1 - p) \frac{1}{2} \lambda e^{-\lambda |z+1|} + p \frac{1}{2} \lambda e^{-\lambda |z-1|}} \]

\[ = \frac{p e^{-\lambda |z-1|}}{(1 - p) e^{-\lambda |z+1|} + p e^{-\lambda |z-1|}} \]

\[ = \frac{p e^{-\lambda |z-1|}}{(1 - p) e^{-\lambda (|z+1|-|z-1|)} + p} \]

The final manipulations are to ease interpretations for \( p \to 0^+ \), \( p \to 1^- \), \( \lambda \to 0^+ \), and \( \lambda \to \infty \). Easily

\[ \lim_{p \to 0^+} P(X = 1 \mid Z = z) = 0 \]

and

\[ \lim_{p \to 1^-} P(X = 1 \mid Z = z) = 1; \]

these make sense because the observation \( z \) should become unimportant when value of \( X \) becomes certain without it. Next,

\[ \lim_{\lambda \to 0^+} P(X = 1 \mid Z = z) = p, \]

which makes sense because the distribution of \( Y \) becomes very flat as \( \lambda \to 0^+ \), making the observation uninformative. Finally,

\[ \lim_{\lambda \to \infty} P(X = 1 \mid Z = z) = \begin{cases} 1, & \text{if } |z+1| > |z-1|, \\ 0, & \text{if } |z+1| < |z-1|, \end{cases} = \begin{cases} 1, & \text{if } z > 0, \\ 0, & \text{if } z < 0; \end{cases} \]

this makes sense because \( \lambda \to \infty \) makes the \( Y \) negligible.

2. We need to apply the version of Bayes rule for a continuous random variable conditioned on a discrete random variable:

\[ f_{Q|X}(q \mid x) = \frac{f_Q(q) p_{X|Q}(x \mid q)}{p_X(x)} = \frac{f_Q(q) p_{X|Q}(x \mid q)}{\int_0^1 f_Q(q) p_{X|Q}(x \mid q) dq} \]

For \( x = 0 \) and \( q \in [0, 1] \),

\[ f_{Q|X}(q \mid 0) = \frac{f_Q(q) p_{X|Q}(0 \mid q)}{\int_0^1 f_Q(q) p_{X|Q}(0 \mid q) dq} = \frac{6q(1-q) \cdot (1-q)}{\int_0^1 6q(1-q)(1-q) dq} = \frac{6q(1-q) \cdot (1-q)}{1/2} = 12q(1-q)^2. \]
For $x = 1$ and $q \in [0, 1]$, 
\[
 f_{Q|X}(q \mid 1) = \frac{f_{Q}(q)p_{X|Q}(1 \mid q)}{\int_{0}^{1} f_{Q}(q)p_{X|Q}(1 \mid q) \, dq} = \frac{6q(1-q) \cdot q}{\int_{0}^{1} 6q(1-q)q \, dq} 
\]
\[
 = \frac{6q(1-q) \cdot q}{1/2} = 12q^2(1-q).
\]

The distributions $f_{Q}(q)$, $f_{Q|X}(q \mid 0)$, and $f_{Q|X}(q \mid 1)$ are all in the family of beta distributions, which arise again in Chapter 8.

3. Because of the definition of $g$, the random variable $Y$ takes on only nonnegative values. Thus $f_{Y}(y) = 0$ for any negative $y$. For $y > 0$,
\[
 F_{Y}(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(X \in [-y, 0]) + \mathbf{P}(X \in (0, y^2])
\]
\[
 = (F_{X}(0) - F_{X}(-y)) + (F_{X}(y^2) - F_{X}(0))
\]
\[
 = F_{X}(y^2) - F_{X}(-y).
\]

Taking the derivative of $F_{Y}(y)$ (and using the chain rule),
\[
 f_{Y}(y) = 2yf_{X}(y^2) + f_{X}(-y)
\]
\[
 = \frac{1}{\sqrt{2\pi}} \left( 2ye^{-y^4/2} + e^{-y^2/2} \right).
\]