Recitation 13 Solutions
October 21, 2010

1. (a) We begin by writing the definition for \( E[Z \mid X, Y] \)

\[
E[Z \mid X = x, Y = y] = \sum_z z p_{Z \mid X, Y}(z \mid x, y)
\]

Since \( E[Z \mid X, Y] \) is a function of the random variables \( X \) and \( Y \), and is equal to \( E[Z \mid X = x, Y = y] \) whenever \( X = x \) and \( Y = y \), which happens with probability \( p_{X,Y}(x, y) \), using the expected value rule, we have

\[
E[E[Z \mid X, Y]] = \sum_x \sum_y E[Z \mid X = x, Y = y] p_{X,Y}(x, y)
\]

\[
= \sum_x \sum_y \sum_z z p_{Z \mid X, Y}(z \mid x, y) p_{X,Y}(x, y)
\]

\[
= \sum_x \sum_y \sum_z z p_{X,Y,Z}(x, y, z)
\]

\[
= E[Z]
\]

(b) We start with the definition for \( E[Z \mid X, Y] \) which is a function of the random variables \( X \) and \( Y \), and is equal to \( E[Z \mid X = x, Y = y] \) whenever \( X = x \) and \( Y = y \), so

\[
E[Z \mid X = x, Y = y] = \sum_z z p_{Z \mid X, Y}(z \mid x, y)
\]

Proceeding as above, but conditioning on the event \( X = x \), we have

\[
E[E[Z \mid X, Y = y] \mid X = x] = \sum_y E[Z \mid X = x, Y = y] p_{Y \mid X}(y \mid x)
\]

\[
= \sum_y \sum_z z p_{Z \mid X, Y}(z \mid x, y) p_{Y \mid X}(y \mid x)
\]

\[
= \sum_y \sum_z z p_{Y,Z \mid X}(y, z \mid x)
\]

\[
= E[Z \mid X = x]
\]

Since this is true for all possible values of \( x \), we have \( E[E[Z \mid Y, X] \mid X] = E[Z \mid X] \).

(c) We take expectations of both sides of the formula in part (b) to obtain

\[
E[E[Z \mid X]] = E[E[E[Z \mid X, Y] \mid X]]
\]

By the law of iterated expectations, the left-hand side above is \( E[Z] \), which establishes the desired result.

2. Let \( Y \) be the length of the piece after we break for the first time. Let \( X \) be the length after we break for the second time.
(a) The law of iterated expectations states:

\[ E[X] = E[E[X|Y]] \]

We have \( E[X|Y] = \frac{Y}{2} \) and \( E[Y] = \frac{1}{2} \). So then:

\[ E[X] = E[E[X|Y]] = E[Y/2] = \frac{1}{2} E[Y] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \]

(b) We use the Law of Total Variance to find \( \text{var}(X) \):

\[ \text{var}(X) = E[\text{var}(X \mid Y)] + \text{var}(E[X \mid Y]) \]

Recall that the variance of a uniform random variable distributed over \([a, b]\) is \((b - a)^2 / 12\). Since \( Y \) is uniformly distributed over \([0, \ell]\), we have

\[ \text{var}(Y) = \frac{\ell^2}{12}, \]
\[ \text{var}(X \mid Y) = \frac{Y^2}{12}. \]

We know that \( E[X \mid Y] = Y/2 \), and so

\[ \text{var}(E[X \mid Y]) = \text{var}(Y/2) = \frac{1}{4} \text{var}(Y) = \frac{\ell^2}{48}. \]

Also,

\[ E[\text{var}(X \mid Y)] = E \left[ \frac{Y^2}{12} \right] \]
\[ = \int_0^\ell \frac{y^2}{12} f_Y(y) dy \]
\[ = \frac{1}{12} \cdot \frac{1}{\ell} \int_0^\ell y^2 dy \]
\[ = \frac{\ell^2}{36}. \]

Combining these results, we obtain

\[ \text{var}(X) = E[\text{var}(X \mid Y)] + \text{var}(E[X \mid Y]) = \frac{\ell^2}{36} + \frac{\ell^2}{48} = \frac{7\ell^2}{144}. \]

3. Let \( X_i \) denote the number of widgets in the \( i \)th box. Then \( T = \sum_{i=1}^N X_i \).

\[ E[T] = E[E[\sum_{i=1}^N X_i \mid N]] \]
\[ = E[\sum_{i=1}^N E[X_i \mid N]] \]
\[ = E[\sum_{i=1}^N E[X]] \]
\[ = E[X] \cdot E[N] = 100. \]
and,

\[
\text{var}(T) = E[\text{var}(T|N)] + \text{var}(E[T|N]) = E\left[\text{var}\left(\sum_{i=1}^{N} X_i|N\right)\right] + \text{var}\left(E\left[\sum_{i=1}^{N} X_i|N\right]\right) = E[N\text{var}(X)] + \text{var}(N E[X]) = (\text{var}(X))E[N] + (E[X])^2 \text{var}(N) = 16 \cdot 10 + 100 \cdot 16 = 1760.
\]