Recitation 14 Solutions
October 26, 2010

1. (a) Let $X$ (time between successive mosquito bites) = (time until the next mosquito bite). The mosquito bites occur according to a Bernoulli process with parameter $p = 0.5 - 0.2 = 0.1$. $X$ is a geometric random variable, so, $E[X] = \frac{1}{p} = \frac{1}{0.1} = 10$.

$$\text{var}(X) = \frac{1 - p}{p^2} = \frac{1 - 0.1}{0.1^2} = 90.$$

(b) Mosquito bites occur according to a Bernoulli process with parameter $p = 0.1$. Tick bites occur according to another independent Bernoulli process with parameter $q = 0.1 \cdot 0.7 = 0.07$. Bug bites (mosquito or tick) occur according to a merged Bernoulli process from the mosquito and tick processes. Therefore, the probability of success at any time point for the merged Bernoulli process is $r = p + q - pq = 0.1 + 0.07 - 0.1 \cdot 0.07 = 0.163$. Let $Y$ be the time between successive bug bites. As before, $Y$ is a geometric random variable, so $E[Y] = \frac{1}{r} = \frac{1}{0.163} \approx 6.135$.

$$\text{var}(Y) = \frac{1 - r}{r^2} = \frac{1 - 0.163}{0.163^2} \approx 31.503$$

2. (a) In this case, since the trials are independent, the given information is irrelevant. $P$(next 2 trials result in 3 tails) = $\left(\frac{1}{8}\right)^2 = \frac{1}{64}$.

(b) i. The second order Pascal PMF for random variable $N$, as defined in the text, is the probability of the second success comes on the $n^{th}$ trial. Thus, the random variable, $K$, is a shifted version of the second order Pascal PMF, i.e. $K = N - 1$. So, the probability that 1 success comes in the first $k$ trials, where the next trial will result in the second success, can be expressed as:

$$p_K(k) = \binom{k}{1} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{k-1}, \quad k \geq 1.$$

ii. The number of tails before the first success, $M$, can be written as a random sum:

$$M = X_1 + X_2 + \cdots + X_N,$$

where $X_i$ is the number of tails that occur on (unsuccessful) trial $i$, and $N$ is the number of unsuccessful trials (i.e. trials before the first success). We notice that $X$ is equally likely to be either 1 or 2, and that $N$ is a shifted geometric: $N = R - 1$, where $R$ is a geometric random variable with parameter $\frac{1}{4}$. Now we can apply our random sum formulae.

$$E[M] = E[X]E[N] = \left(\frac{3}{2}\right)(4 - 1) = \frac{9}{2}$$

$$\text{var}(M) = E[N\text{var}(X) + (E[X])^2\text{var}(N) = (4 - 1)(\frac{1}{4}) + \left(\frac{3}{2}\right)^2(12) = \frac{111}{4}.$$
(c) \( N \), the number of trials in Bob’s experiment, can be expressed as the sum of 3 independent random variables, \( X, Y, \) and \( Z \). \( X \) is the number of trials until Bob removes the first coin, \( Y \) the number of additional trials until he removes the second coin, and \( Z \) the additional number until he removes the third coin. We see that \( X \) is a geometric random variable with parameter \( \frac{1}{8} \), \( Y \) is geometric with parameter \( \frac{1}{4} \), and \( Z \) geometric with parameter \( \frac{1}{2} \). Hence,

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3. Let \( M \) be the total number of draws you make until you have signed all \( n \) papers. Let \( T_i \) be the number of draws you make until drawing the next unsigned paper after having signed \( i \) papers. Then \( M = T_0 + \cdots + T_{n-1} \).

We can view the process of selecting the next unsigned paper after having signed \( i \) papers as a sequence of independent Bernoulli trials with probability of success \( p_i = \frac{n-i}{n} \), since there are \( n-i \) unsigned papers out of a total of \( n \) papers and receiving any paper is equally likely in a particular draw. The PMF governing the number of attempts we make until we succeed in drawing the next unsigned paper after having signed \( i \) papers is geometric. More concretely, the probability that it takes \( k \) tries to draw the next unsigned paper after having signed \( i \) papers is

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P(T_i = k) = (1 - p_i)^{k-1} p_i.
\]

With this model, the expected value of \( M \), the number of draws you make until you sign all \( n \) papers is:

\[
\mathbb{E}[M] = \mathbb{E} \left[ \sum_{i=0}^{n-1} T_i \right] = \sum_{i=0}^{n-1} \mathbb{E}[T_i] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{k=1}^{n} \frac{1}{k}.
\]

For large \( n \), this is on the order of: \( n \int_{1}^{n} \frac{1}{x} \, dx = n \log n \).