LECTURE 3

- **Readings:** Section 1.5
- Review
  - Independence of two events
  - Independence of a collection of events

**Review**

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{assuming } P(B) > 0 \]

- Multiplication rule:
  \[ P(A \cap B) = P(B) \cdot P(A \mid B) = P(A) \cdot P(B \mid A) \]

- Total probability theorem:
  \[ P(B) = P(A)P(B \mid A) + P(A^c)P(B \mid A^c) \]

- Bayes rule:
  \[ P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(B)} \]

**Models based on conditional probabilities**

- 3 tosses of a biased coin:
  \[ P(H) = p, \ P(T) = 1 - p \]

**Independence of two events**

- **Defn:** \( P(B \mid A) = P(B) \)
  - “occurrence of \( A \) provides no information about \( B \)’s occurrence”

- Recall that \( P(A \cap B) = P(A) \cdot P(B \mid A) \)

- **Defn:** \( P(A \cap B) = P(A) \cdot P(B) \)

- Symmetric with respect to \( A \) and \( B \)
  - applies even if \( P(A) = 0 \)
  - implies \( P(A \mid B) = P(A) \)

**Conditioning may affect independence**

- Conditional independence, given \( C \), is defined as independence under probability law \( P(\cdot \mid C) \)

- Assume \( A \) and \( B \) are independent

- If we are told that \( C \) occurred, are \( A \) and \( B \) independent?
Conditioning may affect independence

- Two unfair coins, $A$ and $B$:
  \[ P(H \mid \text{coin } A) = 0.9, \quad P(H \mid \text{coin } B) = 0.1 \]

  Choose either coin with equal probability.

- Once we know it is coin $A$, are tosses independent?
- If we do not know which coin it is, are tosses independent?
  
  - Compare:
    \[ P(\text{toss } 11 = H) \]
    \[ P(\text{toss } 11 = H \mid \text{first 10 tosses are heads}) \]

Independence of a collection of events

- Intuitive definition:
  Information on some of the events tells us nothing about probabilities related to the remaining events
  
  - E.g.:
    \[ P(A_1 \cap (A_2 \cup A_3) \mid A_5 \cap A_6^c) = P(A_1 \cap (A_2 \cup A_3)) \]

- Mathematical definition:
  Events $A_1, A_2, \ldots, A_n$ are called independent if:
  \[ P(A_i \cap A_j \cap \cdots \cap A_q) = P(A_i)P(A_j)\cdots P(A_q) \]
  for any distinct indices $i, j, \ldots, q$, (chosen from $\{1, \ldots, n\}$)

Independence vs. pairwise independence

- Two independent fair coin tosses
  - $A$: First toss is $H$
  - $B$: Second toss is $H$
  - $P(A) = P(B) = 1/2$

  \[
  \begin{array}{cc}
  \text{HH} & \text{HT} \\
  \text{TH} & \text{TT} \\
  \end{array}
  \]

  - $C$: First and second toss give same result
  - $P(C) =$
  - $P(C \cap A) =$
  - $P(A \cap B \cap C) =$
  - $P(C \mid A \cap B) =$

  - Pairwise independence does not imply independence

The king's sibling

- The king comes from a family of two children. What is the probability that his sibling is female?