LECTURE 7

Review

\[ p_X(x) = P(X = x) \]
\[ p_{X,Y}(x, y) = P(X = x, Y = y) \]
\[ p_{X|Y}(x | y) = P(X = x | Y = y) \]
\[ p_X(x) = \sum_y p_{X,Y}(x, y) \]
\[ p_{X,Y}(x, y) = p_X(x)p_{Y|X}(y | x) \]

Independent random variables

\[ p_{X,Y,Z}(x, y, z) = p_X(x)p_{Y|X}(y | x)p_{Z|X,Y}(z | x, y) \]

- Random variables \( X, Y, Z \) are independent if:
  \[ p_{X,Y,Z}(x, y, z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z) \]
  for all \( x, y, z \)

- Independent?
  - What if we condition on \( X \leq 2 \) and \( Y \geq 3? \)

Expectations

\[ E[X] = \sum_x x p_X(x) \]
\[ E[g(X,Y)] = \sum_x \sum_y g(x, y)p_{X,Y}(x, y) \]

- In general: \( E[g(X,Y)] \neq g(E[X], E[Y]) \)

- \( E[\alpha X + \beta] = \alpha E[X] + \beta \)

- If \( X, Y \) are independent:
  - \( E[XY] = E[X]E[Y] \)
  - \( E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)] \)
Variances

- $\text{Var}(aX) = a^2 \text{Var}(X)$
- $\text{Var}(X + a) = \text{Var}(X)$

- Let $Z = X + Y$.
  If $X, Y$ are independent:
  \[
  \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)
  \]

- Examples:
  - If $X = Y$, $\text{Var}(X + Y) = $
  - If $X = -Y$, $\text{Var}(X + Y) = $
  - If $X, Y$ indep., and $Z = X - 3Y$, $\text{Var}(Z) = $

**Binomial mean and variance**

- $X =$ # of successes in $n$ independent trials
  - probability of success $p$
    \[
    E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}
    \]
  - $X_i = \begin{cases} 1, & \text{if success in trial } i, \\ 0, & \text{otherwise} \end{cases}$

- $E[X_i] = $
- $E[X] = $
- $\text{Var}(X_i) = $
- $\text{Var}(X) = $

**The hat problem**

- $n$ people throw their hats in a box and then pick one at random.
  - $X$: number of people who get their own hat
  - Find $E[X]$

  \[
  X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}
  \]

- $X = X_1 + X_2 + \cdots + X_n$
- $P(X_i = 1) = $
- $E[X_i] = $
- Are the $X_i$ independent?
- $E[X] = $

**Variance in the hat problem**

- $\text{Var}(X) = E[X^2] - (E[X])^2 = E[X^2] - 1$

  \[
  X^2 = \sum_i X_i^2 + \sum_{i,j:i\neq j} X_i X_j
  \]

- $E[X^2] = $

  \[
  P(X_1 X_2 = 1) = P(X_1 = 1) \cdot P(X_2 = 1 \mid X_1 = 1)
  \]

  =

- $E[X^2] = $
- $\text{Var}(X) = $