LECTURE 3

• Readings: Sections 1.5

Lecture outline

• Review
• Independence of two events
• Independence of a collection of events
Review

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ assuming } P(B) > 0. \]

- Multiplication rule:
  \[ P(A \cap B) = P(B) \cdot P(A \mid B) = P(A) \cdot P(B \mid A) \]

- Total probability theorem:
  \[ P(B) = P(A)P(B \mid A) + P(A^c)P(B \mid A^c) \]

- Bayes rule:
  \[ P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(B)} \]
Extended Radar Example

- Threat alert affects the outcome

<table>
<thead>
<tr>
<th>Radar</th>
<th>Low(0)</th>
<th>Medium(?)</th>
<th>High(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airplane</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Absent</td>
<td>0.1125</td>
<td>0.05</td>
<td>0.0125</td>
</tr>
<tr>
<td>Present</td>
<td>0.055</td>
<td>0.22</td>
<td>0.55</td>
</tr>
</tbody>
</table>

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<tr>
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<td>0.20</td>
<td>0.05</td>
</tr>
<tr>
<td>Present</td>
<td>0.02</td>
<td>0.08</td>
<td>0.20</td>
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\[P(\text{Threat})=\text{Prior probability of threat}= \ p\]
Extended Radar Example  
(continued)

• A=Airplane, R=Radar Reading

\[
P(A, R) = P(\text{Threat})P(A, R|\text{Threat}) + P(\text{No Threat})P(A, R|\text{No Threat})
\]

• If we let \( p = P(\text{Threat}) \), then we get:

<table>
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<tr>
<td>Absent</td>
<td>0.45-0.3375(p)</td>
<td>0.20-0.15(p)</td>
<td>0.05-0.0375(p)</td>
<td></td>
</tr>
<tr>
<td>Present</td>
<td>0.02+0.014(5p)</td>
<td>0.08+0.14(p)</td>
<td>0.20+35(p)</td>
<td></td>
</tr>
</tbody>
</table>
Extended Radar Example
(continued)

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<td>0.20-</td>
<td>0.05-</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.3375p</td>
<td>0.15p</td>
<td>0.0375p</td>
<td></td>
</tr>
<tr>
<td>Present</td>
<td>0.02+</td>
<td>0.08+</td>
<td>0.20+</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.014</td>
<td>0.14p</td>
<td>35p</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5p</td>
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- Given the Radar registered High, and a plane was absent, What is the probability that there was a threat?
- How does the decision region behave, as a function of $p$?
Independence of Two Events

• Definition: \( P(A \cap B) = P(A) \cdot P(B|A) \)

• Recall:
  – Independence of \( B \) from \( A \):
    \[ P(B|A) = P(B) \]
  – By symmetry, \( P(A|B) = P(A) \)

• Examples:
  – \( A \) and \( B \) are disjoint.
  – Independence of \( A^c \) and \( B \).
  – \( P(A|B) = P(A|B^c) \)
Conditioning may affect independence

• Assume $A$ and $B$ are independent:

• If we are told that $C$ occurred, are $A$ and $B$ independent?
Conditioning may affect independence

- **Example 1:**
  - Two independent fair \((p = \frac{1}{2})\) coin tosses.
  - Event \(A\): First toss is H
  - Event \(B\): Second toss is H
  - \(P(A) = P(B) = \frac{1}{2}\)
  - Event \(C\): The two outcomes are different.
  - Conditioned on \(C\), are \(A\) and \(B\) independent?
Conditioning may affect independence

• Example 2:
  – Choice between two unfair coins, with equal probability.
  – \( P(H|\text{coin 1}) = 0.9 \),  
    \( P(H|\text{coin 2}) = 0.1 \)
  – Keep tossing the chosen coin.

• Are future tosses independent:
  – If we know we chose coin A?
  – If we do not know which coin we chose?
  – Compare: \( P(\text{toss 11} = H) \)  
    \( P(\text{toss 11} = H| \text{first 10 tosses are H}) \)
Independence of a Collection of Events

• Intuitive definition:
  – Information about some of the events tells us nothing about probabilities related to remaining events.

  – Example: $P(A_1 \cap (A_2^c \cup A_3) \mid A_5 \cap A_6^c) = P(A_1 \cap (A_2^c \cup A_3))$

• Mathematical definition:
  – For any distinct $i, j, \ldots, q$:

    $$P(A_i \cap A_j \cap \cdots \cap A_q) = P(A_i)P(A_j)\cdots P(A_q)$$
Independence vs. Pairwise Independence

- Example 1 Revisited:
  - Two independent fair \((p = \frac{1}{2})\) coin tosses.
  - Event \(A\): First toss is H
  - Event \(B\): Second toss is H
  - Event \(C\): The two outcomes are different.

- \(P(C) = P(A) = P(B) = \frac{1}{2}\)
- \(P(C \cap A) = \frac{1}{4}\)
- \(P(C \mid A \cap B) = 0\)

- Pairwise independence does not imply independence.
The King’s Sibling

• The king comes from a family of two children.
• What is the probability that his sibling is female?