LECTURE 7

• Readings: Finish Chapter 2

Lecture outline

• Joint PMFs
• Independent random variables
• More expectations, variances
• Binomial distribution revisited
• The hat problem
• Application: Point-to-Point Communication
Review

- Random Variables and PMF
- Expectation
- Variance
- Examples:
  - Binomial, Geometric, and Poisson
Joint PMFs

- \( p_{X,Y}(x, y) = P(X = x \text{ and } Y = y) \)

- \( p_X(x) = \sum_y p_{X,Y}(x, y) \)

- \( p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \)

- \( \sum_x \sum_y p_{X,Y}(x, y) = 1 \)

- \( \sum_x p_{X|Y}(x|y) = 1 \)
Independent Random Variables

\[ p_{X,Y,Z}(x, y, z) = p_X(x)p_{Y|X}(y|x)p_{Z|X,Y}(z|x, y) \]

- Random variables \( X, Y \) and \( Z \) are independent if (for all \( x, y \) and \( z \)):

\[ p_{X,Y,Z}(x, y, z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z) \]

- Example:
  Independent?

- What if we condition on \( X \leq 2 \) and \( Y \geq 3 \)?
Expectations

\[ E[X] = \sum_x x \cdot p_X(x) \]

\[ E[g(X, Y)] = \sum_x \sum_y g(x, y) \cdot p_{X,Y}(x, y) \]

- In general: \( E[g(X, Y)] \neq g(E[X], E[Y]) \)
- \( E[\alpha X + \beta] = \alpha E[X] + \beta \)
- If \( X \) and \( Y \) are independent:
  - \( E[X \cdot Y] = E[X] \cdot E[Y] \)
  - \( E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)] \)
Variances

• \( \text{var}(aX) = a^2 \text{var}(X) \)
• \( \text{var}(X + a) = \text{var}(X) \)

• Let \( Z = X + Y \). If \( X \) and \( Y \) independent:
  \[
  \text{var}(X + Y) = \text{var}(X) + \text{var}(Y)
  \]

• Examples:
  
  – If \( X = Y \), \( \text{var}(X + Y) = 4 \text{var}(X) \)
  
  – If \( X = -Y \), \( \text{var}(X + Y) = 0 \)
  
  – If \( X, Y \) indep., and \( Z = X - 3Y \), \( \text{var}(Z) = \text{var}(X) + 9 \text{var}(Y) \)
Binomial Mean and Variance

- $X = \#$ of successes in $n$ independent trials
  - Probability of success: $p$

  $$E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1 - p)^{n-k}$$

- $X_i = \begin{cases} 1, & \text{if success in trial } i, \\ 0, & \text{otherwise} \end{cases}$

- $E[X_i] = p$
- $\text{var}(X_i) = p - p^2$

- $E[X] = np$
- $\text{var}(X) = np(1 - p)$
The Hat Problem

- \( n \) people throw their hats in a box and then pick one at random.
  - \( X \): number of people who get their own hat
  - Find \( \mathbb{E}[X] \)

\[
X_i = \begin{cases} 
1, & \text{if } i \text{ selects own hat,} \\
0, & \text{otherwise.}
\end{cases}
\]

- \( X = X_1 + X_2 + \cdots + X_n \)
- \( \mathbb{P}(X_i = 1) = \frac{1}{n} \)
- \( \mathbb{E}[X_i] = \frac{1}{n} \)
- Are the \( X_i \) independent? \( \text{No} \)
- \( \mathbb{E}[X] = n(\frac{1}{n}) = 1 \)
Variance in the Hat Problem

- \( \text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[X^2] - 1 \)

\[ X^2 = \sum_i X_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_i X_j \]

- \( \mathbb{E}[X_i^2] = \frac{1}{n} \)

\[ P(X_1 X_2 = 1) = P(X_1 = 1) \cdot P(X_2 = 1 | X_1 = 1) = \left( \frac{1}{n} \right) \left( \frac{1}{n-1} \right) \]

- \( \mathbb{E}[X^2] = n\frac{1}{n} + n(n-1)\left( \frac{1}{n} \right) \left( \frac{1}{n-1} \right) = 2 \)

- \( \text{var}(X) = 1 \)
Challenge: BBall Party

Your Guests are all BBall fans and they wear BBall Caps. There is a total of $s$ teams in the league. Everyone of your guests is equally likely to be a fan of any one of these teams.

Compute the expected number of people who will pick a cap from their own team!
A Communication Example

Introduction

- A point-to-point communication system.

- Probabilistic model:
  - Messages are independent binary r.v.s.
  - The encoder is a deterministic function.
  - The channel introduces errors. It is modeled as a conditional pmf.
  - The decoder is a deterministic function.
A Communication Example
Messages, Encoding

• **Messages**: I.I.D. Bernoulli r.v.s \( M_1, M_2, \ldots \)

\[
M_i = \begin{cases} 
1, & \text{with probability } p \\
0, & \text{with probability } 1 - p 
\end{cases}
\]

• **Encoding**: Repeat \( n \) times,

\[
\{0, 1\} \longleftrightarrow \{0, 1\}^n
\]

\[
0 \quad \rightarrow \quad 00 \cdots 0 \\
1 \quad \rightarrow \quad 11 \cdots 1
\]
A Communication Example
Channel - 1

- Encoded bits are transmitted independently one by one through the channel.
- The channel flips each bit independently, and with "crossover" probability $e$.
- Pictorially:
A Communication Example

Channel - 2

- Mathematically:

\[ p_{Y|X}(y|x) = \begin{cases} 
1 - e & \text{If } x = y, \\
e & \text{If } x \neq y. 
\end{cases} \]

- Multiple transmissions:

\[ Y_1, Y_2, \cdots \]

\[ X_1, X_2, \cdots \]

\[
\mathbb{P}_{Y_1, Y_2, \cdots |X_1, X_2, \cdots}(y_1, y_2, \cdots | x_1, x_2, \cdots) = \mathbb{P}_{Y|X}(y_1|x_1) \cdot \mathbb{P}_{Y|X}(y_2|x_2) \cdots
\]
### A Communication Example

**Decoding**

- **Decoding**: Majority Rule
  - Consider a single message: \( M \)
  - Encoded r.v.s: \( X_1, \ldots, X_n \)
  - Received r.v.s: \( Y_1, \ldots, Y_n \)
  - Decoded message is a function of \( Y_1, \ldots, Y_n \):

\[
\widehat{M}_{Y_1, \ldots, Y_N}(y_1, \cdots, y_n) = \begin{cases} 
0 & \text{If } y_1 + \cdots + y_n < n/2, \\
1 & \text{If } y_1 + \cdots + y_n \geq n/2.
\end{cases}
\]
A Communication Example

Performance

- If $n = 1$, what is $P(\hat{M} \neq M)$?
- What if $n = 3$?
- What if $n$ is made arbitrarily large?
- Is there anything lost?
- How good is the decision rule?
A Communication Example

Probability of Error

Source Messages → Encoder → Channel → Decoder → Received Messages

- **Probability of error:**
  \[
  P(\hat{M} \neq M) = P(\hat{M} = 1|M = 0)(1 - p) + P(\hat{M} = 0|M = 1)p
  \]

  \[
  P(\hat{M} = 1|M = 0) = P(Y_1 + \cdots + Y_n > \frac{n}{2}|M = 0)
  \]

  \[
  = \sum_{k \geq \frac{n}{2}} \binom{n}{k} e^k (1 - e)^{n-k}
  \]