LECTURE 16

• Readings: Section 5.1

Lecture outline

• Random processes
• Definition of the Bernoulli process
• Basic properties of the Bernoulli process
  – Number of successes
  – Distribution of interarrival times
  – The time of the $k^{th}$ success
Random Processes: Motivation

- Sequence of random variables: \( X_1, X_2, \cdots \)

Examples:

- **Arrival example**: Arrival of people to a bank.
- **Queuing example**: Length of a line at a bank.
- **Gambler’s ruin**: The probability of an outcome is a function of the probability of other outcomes (Markov Chains).
- **Engineering example**: Signal corrupted with noise.
The Bernoulli Process

- A sequence of independent Bernoulli trials.
- At each trial:
  - $P(\text{success}) = P(X = 1) = p$
  - $P(\text{failure}) = P(X = 0) = 1 - p$

Examples:
- Sequence of ups and downs of the Dow Jones.
- Sequence of lottery wins/losses.
- Arrivals (each second) to a bank.
Number of successes $S$ in $n$ time slots

- $1 \ 2 \ \cdots \ n$
- $k = 6$

$\mathbb{P}(S = k) = \binom{n}{k} p^k (1 - p)^{n-k}$, \hspace{1cm} \text{(Binomial)}

- **Mean:** $\mathbb{E}[S] = np$

- **Variance:** $\text{Var}(S) = np(1 - p)$
Interarrival Times

• $T_1$: number of trials until first success (inclusive).

• $P(T_1 = t) = p(1 - p)^{t-1}$, \hspace{1cm} (Geometric)
  \hspace{1cm} t = 1, 2, \ldots$

• **Memoryless property.**

• **Mean:** $E[T_1] = \frac{1}{p}$

• **Variance:** $\text{Var}(T_1) = \frac{1 - p}{p^2}$
Fresh Start and Memoryless Properties

**Fresh Start**
Given $n$, the future sequence $X_{n+1}, X_{n+2}, \ldots$ is also a Bernoulli process and is independent of the past.

**Memorylessness**
Suppose we observe the process for $n$ times and no success occurred. Then the pmf of the remaining time for arrival is geometric.

$$P(T - n = k \mid T > n) = p(1 - p)^{k-1}$$
Time of the $k^{th}$ Arrival

$t = 15$

- $Y_k$: number of trials until $k^{th}$ success (inclusive).
- $T_k = Y_k - Y_{k-1}$, $k = 2, 3, \ldots$: $k$th interarrival time

It follows that:

$$Y_k = T_1 + T_2 + \ldots + T_k$$
Time of the $k^{th}$ Arrival

$1 \ 2 \ \ldots \ k = 4 \ \ n$

t = 15

• $Y_k$: number of trials until $k^{th}$ success (inclusive).

• Mean: $\mathbb{E}[Y_k] = \frac{k}{p}$

• Variance: $\text{Var}(Y_k) = \frac{k(1-p)}{p^2}$

• $P(Y_k = t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}$, \hspace{1cm} (Pascal)

t = k, k + 1, \ldots