LECTURE 18

• Readings: Finish Section 5.2

Lecture outline

• Review of the Poisson process
• Properties
  – Adding Poisson Processes
  – Splitting Poisson Processes
• Examples
The Poisson Process: Review

- Number of arrivals in disjoint time intervals are independent, $\lambda = \text{“arrival rate”}$

$$P(k, \delta) \approx \begin{cases} 
1 - \lambda \delta & \text{if } k = 0 \\
\lambda \delta & \text{if } k = 1 \\
0 & \text{if } k > 0 
\end{cases} \quad \text{(for very small } \delta)$$

$$P(k, \tau) = \frac{(\lambda \tau)^k}{k!} e^{-\lambda \tau} \quad \mathbf{E}(N) = \lambda \tau \quad \text{(Poisson)}$$

- Interarrival times ($k = 1$):

$$f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0 \quad \text{(Exponential)}$$

- Time to the $k^{th}$ arrival:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0 \quad \text{(Erlang)}$$
Example: **Poisson Catches**

- Catching fish according to Poisson $\lambda = 0.6$/hour.
- Fish for two hours, but if there’s no catch, continue until the first one.

\[
\begin{align*}
\mathbb{P}(\text{fish more than 2 hrs}) &= \\
\mathbb{P}(\text{fish more than 2 but less than 5 hrs}) &= \\
\mathbb{P}(\text{catch at least 2 fish}) &=
\end{align*}
\]
Example: **Poisson Catches**

- Catching fish according to Poisson $\lambda = 0.6$/hour.
- Fish for two hours, but if there’s no catch, continue until the first one.

\[ E[\text{number of fish}] = \]

\[ E[\text{future fishing time} | \text{fished for 4 hrs}] = \]

\[ E[\text{total fishing time}] = \]
Adding (Merging) Poisson Processes

• Sum of independent Poisson random variables is Poisson.

• Sum of independent Poisson processes is Poisson.

• What is the probability that the next arrival comes from the first process?

$$\frac{\lambda_1 \delta}{\lambda_1 \delta + \lambda_2 \delta} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$
Splitting of Poisson Processes

- Each message is routed along the first stream with probability $p$, and along the second stream with probability $1 - p$.
  - Routing of different messages are independent.
  - Each output stream is Poisson.

\[
\begin{align*}
\lambda_1 &= p \cdot \lambda \\
\lambda_2 &= (1 - p) \cdot \lambda
\end{align*}
\]
Example: **Email Filter** (1)

- You have incoming email from two sources: valid email, and spam. We assume both to be Poisson.
- Your receive, on average, 2 valid emails per hour, and 1 spam email every 5 hours.

![Diagram](image)

- Total incoming email rate = \( \lambda = \lambda_1 + \lambda_2 = 2.2 \) emails per hour.
- Probability that a received email is spam = \( \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{0.2}{2.2} \approx 0.09 \)
Example: Email Filter (2)

- You install a spam filter, that filters out spam email correctly 80% of the time, but also identifies a valid email as spam 5% of the time.

\[ p = 0.95 \quad q = 0.2 \]

- Inbox email rate = \[ p\lambda_1 + q\lambda_2 = 0.95 \cdot 2 + 0.2 \cdot 0.2 = 1.94 \]
- Spam folder email rate = \[ 2.2 - 1.94 = 0.26 \]
Example: **Email Filter (3)**

- Probability that an email in the inbox is spam: 
  \[ p \frac{q \lambda_2}{p \lambda_1 + q \lambda_2} = \frac{0.2 \cdot 0.2}{1.94} \approx 0.02 \]

- Probability that an email in the spam folder is valid:
  \[ \frac{(1-p)\lambda_1}{(1-p)\lambda_1 + (1-q)\lambda_2} = \frac{0.05 \cdot 2}{0.26} \approx 0.38 \]

- Every how often should you check your spam folder, to find one valid email, on average?
  \[ E(N) = \lambda_1 (1-p)\tau = 1 \Rightarrow \tau = \frac{1}{0.05 \cdot 2} = 10 \text{ hrs.} \]