LECTURE 19

• Readings: Finish Section 5.2

Lecture outline

• Markov Processes – I
  – Checkout counter example.
  – Markov process: definition.
  – \( n \)-step transition probabilities.
  – Classification of states.
Example: **Checkout Counter**

- **Discrete time** $n = 0, 1, \ldots$
- **Customer arrivals**: Bernoulli($p$)
  - Geometric interarrival times.
- **Customer service times**: Geometric($q$)
- “State” $X_n$: number of customers at time $n$.
Finite State Markov Models

• $X_n$ : state after $n$ transitions
  - Belongs to a finite set, e.g. $\{1, \cdots, m\}$
  - $X_0$ is either given or random.

• Markov Property / Assumption:
  - Given the current state, the past does not matter.

\[
p_{i,j} = P(X_{n+1} = j | X_n = i, X_{n-1}, \cdots, X_0) \\
= P(X_{n+1} = j | X_n = i)
\]

• Modeling steps:
  - Identify the possible states.
  - Mark the possible transitions.
  - Record the transition probabilities.
$n$-step Transition Probabilities

- State occupancy probabilities, given initial state $i$:
  \[ r_{ij}(n) = P(X_n = j | X_0 = i) \]

- Key recursion:
  \[ r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj} \]

- Random initial state:
  \[ P(X_n = j) = \sum_{i=1}^{m} P(X_0 = i) r_{ij}(n) \]
Example

\[ \begin{array}{cccccc}
 n = 0 & n = 1 & n = 2 & n = 2563 & n = 2564 \\
 r_{11}(n) & 1 & 0.5 & 0.35 & \\
 r_{12}(n) & 0 & 0.5 & 0.65 & 
\end{array} \]
Generic Question

- Does \( r_{ij}(n) \) converge to something?

\[
\begin{align*}
0.5 & \quad 0.5 \\
1 & \quad 1 \\
\end{align*}
\]

\( n \) odd: \( r_{22}(n) = 0 \)

\( n \) even: \( r_{22}(n) = 1 \)

- Does the limit depend on the initial state?

\[
\begin{align*}
0.4 & \\
0.3 & \quad 0.3 \\
\end{align*}
\]

\( r_{11}(n) = 1 \)

\( r_{31}(n) = 0 \)

\[
\begin{align*}
r_{21}(n) &= r_{21}(n - 1) + p_{21}r_{22}(n - 1) \\
&= r_{21}(n - 1) + 0.3(0.4)^{n-1} \\
&= 0.3(1 + 0.4 + \cdots + 0.4^{n-1}) \\
&= 0.5
\end{align*}
\]
Recurrent and Transient States

• State $i$ is **recurrent** if:
  
  - Starting from $i$, and from wherever you can go, there is a way of returning to $i$.

• If not recurrent, a state is called **transient**.
  
  - If $i$ is transient then $\Pr(X_n = i) \to 0$ as $n \to \infty$.
  
  - State $i$ is visited only a finite number of times.

• **Recurrent Class**:
  
  - Collection of recurrent states that “communicate” to each other, and to no other state.
Periodic States

- The states in a recurrent class are \textbf{periodic} if:
  - They can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.

- In this case, $r_{ii}(n)$ cannot converge.