Recitation 5 Solutions  
February 28, 2006

1. Problem 2.22, page 123 in the text. See online solutions.

2. Total expectation follows easily from total probability. This could be a good time to point out that the Total Probability Theorem and Total Expectation Theorem each have versions phrased with (a) conditioning on events forming a partition; and (b) conditioning on a discrete random variable. These are equivalent because the collection of events \( \{Y = y\} \) over all \( y \) is a partition. You could also point out that technically, when we write

\[
E[X] = \sum_y p_Y(y)E[X \mid Y = y]
\]

we better only include in the summation \( y \) such that \( P(Y = y) > 0 \).

3. The result follows by rewriting the expectation summation in the following manner:

\[
E[X] = \sum_{k=0}^\infty kp_X(k) = \sum_{k=1}^\infty \left( \sum_{\ell=1}^{k-1} \right) p_X(k) = \sum_{\ell=1}^\infty \sum_{k=\ell}^\infty p_X(k)
\]

\[
= \sum_{\ell=1}^\infty P(X > \ell - 1) = \sum_{n=0}^\infty P(X > n).
\]

The manipulations could look unmotivated, but if you sketch the \( k-\ell \) plane, then the interchange of summations is clear.