Recitation 09 Answers
March 21, 2006

1. 
   
   (a) \( P[A] = \frac{7}{8} \)
   
   (b) \( P[\text{Al wins 7 out of 10 races}] = \binom{10}{7} \left( \frac{7}{8} \right)^7 \left( \frac{1}{8} \right)^3 \)
   
   (c) \( f_{w_0}(w_0) = \begin{cases} 
   \frac{1}{2}, & 1 < w_0 \leq 2 \\
   \frac{7}{4} - \frac{w_0}{2}, & 2 < w_0 \leq 3 \\
   0, & \text{otherwise} 
   \end{cases} \)

2. 
   
   \[ f_W(w) = \int_{-\infty}^{\infty} f_X(x)f_Y(w-x)dx \]
   
   for \( w = x + y \) and \( x, y \) independent. This operation is called the convolution of \( f_X(x) \) and \( f_Y(y) \).

   \[ f_W(w) = \begin{cases} 
   5w, & 0 \leq w \leq 0.1 \\
   0.5, & 0.1 \leq w \leq 0.9 \\
   5(0.1 + (w - 0.9)), & 0.9 \leq w \leq 1.0 \\
   5(0.1 + (1.1 - w)), & 1.0 \leq w \leq 1.1 \\
   0.5, & 1.1 \leq w \leq 1.9 \\
   5(2.0 - w), & 1.9 \leq w \leq 2.0 \\
   0, & \text{otherwise} 
   \end{cases} \]

3. Let \( X \) and \( Y \) be the number of flips until Alice and Bob stop, respectively. Thus, \( X + Y \) is the total number of flips until both stop. The random variables \( X \) and \( Y \) are independent geometric random variables with parameters \( 1/4 \) and \( 3/4 \), respectively. By convolution, we have

   \[
   p_{X+Y}(j) = \sum_{k=-\infty}^{\infty} p_X(k)p_Y(j-k) \\
   = \sum_{k=1}^{j-1} (1/4)(3/4)^{k-1}(3/4)(1/4)^{j-k-1} \\
   = \frac{1}{4^j} \sum_{k=1}^{j-1} 3^k \\
   = \frac{1}{4^j} \left( \frac{3^j - 1}{3 - 1} - 1 \right) \\
   = \frac{3 (3^{j-1} - 1)}{2 4^j},
   \]
if \( j \geq 2 \), and 0 otherwise. (Even though \( X + Y \) is not geometric, it roughly behaves like one with parameter \( 3/4 \)).