1. a) To find the transform, we integrate the density function over its full domain, against an exponential. This is often expressed as finding the expected value of the function $e^{-rx}$.

$$E[e^{-rx}] = \int_a^b \frac{e^{-rx}}{b-a} \, dx = \frac{e^{-ra} - e^{-rb}}{r(b-a)}.$$ 

b) To find the mean and the variance we use the moment generating properties of the transform, namely:

$$E[X^n] = (-1)^n \frac{d}{dr} E[e^{-rx}] \bigg|_{r=0}$$

Thus we have:

$$E[X] = -\frac{d}{dr} E[e^{-rx}] \bigg|_{r=0} = - \left\{ \left( \frac{1}{b-a} \right) \frac{e^{-rb} - e^{-ra}}{r^2} + \left( \frac{1}{b-a} \right) \frac{be^{-rb} - ae^{-ra}}{r} \right\} \bigg|_{r=0}$$

$$(L'Hôpital) = \frac{b^2 - a^2}{2b-a} - \frac{a^2 - b^2}{b-a} = \frac{b+a}{2}.$$ 

To find the Variance we need to find $E[X^2]$ and thus we need to take the second derivative of the transform and evaluate at $r = 0$,

$$E[X^2] = \frac{d^2}{dr^2} E[e^{-rx}] \bigg|_{r=0} = \left\{ \left( \frac{2}{b-a} \right) \frac{e^{-ra} - e^{-rb}}{r^3} + \left( \frac{2}{b-a} \right) \frac{ae^{-ra} - be^{-rb}}{r^2} + \left( \frac{1}{b-a} \right) \frac{a^2 e^{-ra} - b^2 e^{-rb}}{r} \right\} \bigg|_{r=0}$$

$$(L'Hôpital) = \frac{1}{3} \frac{b^3 - a^3}{b-a} + \frac{a^3 - b^3}{b-a} + \frac{b^3 - a^3}{b-a} = \frac{1}{3} (b^2 + ab + a^2)$$

and therefore we have:

$$Var[X] = E[X^2] - E[X]^2 = \frac{1}{3} (b^2 + ab + a^2) - \left( \frac{b+a}{2} \right)^2$$

2. The transform for nonegative integer valued random variables is defined as:

$$p_x^T(z) = \sum_{i=1}^{\infty} z^{x_i} P(X = x_i) = E[z^X]$$
and therefore we have:

$$E[z^X] = \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{4}z^3.$$

b) We observe from above that if we take $n$ derivatives of the transform and evaluate at $z = 1$ then we will have a linear combination of the first $n$ moments.

$$\frac{d}{dz}E[z^X]_{z=1} = E[X]$$

$$\frac{d^2}{dz^2}E[z^X]_{z=1} = E[X^2] - E[X]$$

$$\frac{d^3}{dz^3}E[z^X]_{z=1} = E[X^3] - 3E[X^2] + 2E[X]$$

and therefore we find:

$$E[X] = \left. \frac{d}{dz}E[z^X] \right|_{z=1} = \frac{1}{2} + \frac{1}{4} + \frac{3}{4} = \frac{7}{4}$$

and similarly,

$$E[X^2] = \left. \frac{d^2}{dz^2}E[z^X] \right|_{z=1} + E[X] = \frac{1}{2} + \frac{3}{4} + \frac{7}{4} = \frac{15}{4}$$

and finally,

$$E[X^3] = \left. \frac{d^3}{dz^3}E[z^X] \right|_{z=1} + 3E[X^2] - 2E[X] = \frac{6}{4} + \frac{45}{4} + \frac{14}{4} = \frac{37}{4}$$

c) Direct computation thankfully produces the same results.

3. a) Note that by the definition of the transform,

$$M_X(s) = \sum_x e^{sx}p_X(x)$$

and therefore when evaluated at $s = 0$, the transform should equal 1. We see that only the second option satisfies this requirement.

b) It is observed that the transform is that of a Poisson random variable with parameter $\lambda = 2$. Hence the pdf is given as follows:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$p_X(0) = e^{-2}$$