

## 6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: A Coin Tossing Puzzle

Hi. In this problem, we'll be going over practice with the calculation of conditional probabilities. We'll start with a game where our friend Alice will be tossing a coin with certain bias of having a head, and tosses this coin twice. And we're interested in knowing, what's the probability that both coin tosses will end up being a head?

The first step we're going to do is to convert the problem into a mathematical form by defining two events as the following. Event A is where the first coin toss is a head. And similarly, event B will be having the second coin toss also being a head. Having these two events will allow us to say, well, the event that A intersection B will be the event that both coin tosses are a head. And we'd like to know the probability of such an event.

In particular, the probability of A and B will be calculated under two types of information. In the first case, we'll be conditioning on that we know the first coin toss is a head. I'd like to know what the probability of A and B is. In the second case, we know that at least one of the two coin tosses is a head expressed in the form A union B.

And under this conditioning, what is the probability of A and B, A intersection B? So Alice, in this problem, says-- well, her guess will be that the first quantity is no smaller than the second quantity. Namely, knowing that the first coin toss is a head somehow more strongly implies that both coin tosses will be a head, compared to the case that we only know at least one of the two coin tosses is a head.

And we'd like to verify if this inequality is indeed true. To do so, let's just use the basic calculation of conditional probability. Now, from the lectures, you've already learned that to calculate this quantity, we'll write out a fraction where the numerator is the probability of the intersection of these two events. So we have A intersect B intersection A divided by the probability of the event that we're conditioning on, which is A.

Now, the top quantity, since we know that A and B is a subset of event A, then taking the intersection of these two quantities will just give us the first event. So we have A and B. And the bottom is still probability of A.

Let's do the same thing for the second quantity here. We have the top probability of A and B intersection the event A union B, and on the bottom, probability of the event A and B. Again, we see the event A and B is a subset of the event A union B. So the top will be A and B. And the bottom-- A union B.

OK, now let's stop for a little bit. We've computed the probability for each expression in the following fractional form. And we observed that for both fractions, the numerator is the same. So the numerator is a probability of A and B. And the denominator in the first case is probably of A, and the second case, probably of A union B.

Since we know that  $A$  is a subset of the event  $A \cup B$ , and by the monotonicity of probabilities, we know that the probability of  $A$  is hence no greater than a probability of  $A \cup B$ . Substituting this back into these expressions, we know that because they lie in the denominators, the first expression is indeed no smaller than the second expression. So our friend Alice was correct.

So throughout this problem, we never used the fact that the probability of a particular coin toss results, let's say, in a head is a certain number. Actually, this bias for the coin is irrelevant. Whether the coin is fair or unfair, this fact is always true.

So indeed, it does not depend on the probability of the coin. But if you're really curious what happens when the coin is fair, we can plug in the numbers. And here, we're assuming the coin is fair, which means probability of having a head is  $1/2$ . Then, we'll see after going through the calculations that the first probability is  $1/2$ , whereas the second probability is  $1/3$ , which means, in this case, the [? dominance ?] actually is strict. So the first one is strictly greater than the second one, OK?

So this completes the first part of the problem. How do we generalize this into more general settings? There are multiple ways, but we'll go over one particular form. And to do so, we'll be defining three events somewhat more abstractly.

Let's say we have three events--  $C$ ,  $D$ , and  $E$ . Imagine any event, but all three events have to satisfy the following condition. First, event  $D$  will be a subset of  $E$ . And second, the intersection of  $C$  and  $D$  is equal to the intersection of  $C$  and  $E$ , OK? So this will be our choice events.

And let's see a particular example. Let's say you have a sample space here and some event  $E$ . Now, by the first condition,  $D$  will have to lie somewhere in  $E$ . For the second condition, we'll pick some event  $C$  such that this is true. And one way to do so is simply picking  $C$  that lies within both  $D$  and  $E$ . And you can see  $C \cap D$  will be  $C$ . And  $C \cap E$  will still be  $C$ . Hence, the second equality is true.

So if both equalities are true, we have the following relationship, that the probability of  $C$  conditional on  $D$  will be no smaller than the probability of  $C$  conditional on event  $E$ . And this will be the more general form of the inequality that we saw before.

So first of all, the way to prove this is in fact the same. We simply write out the value of this using the fractional form. And based on these two facts, we can arrive at this equation, which I shall now go over. But just to see why this form is more general, in fact, if we-- say we let  $C$  be the event  $A \cap B$ ,  $D$  be the event  $A$ , and  $E$  be the event  $A \cup B$  where  $A$  and  $B$  are the events that we defined earlier.

We can verify that, indeed, these conditions are true, namely  $D$  is a subset of  $E$ . Because  $A$  is a subset of  $A \cup B$ , and  $C$  is a subset of both  $D$  and  $E$ . And hence, condition two is also true. And if that's the case, we will actually recover the result we got earlier for events  $A$  and  $B$ . And hence, this equation here is a more general form. So that's the end of the problem. See you next time.

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