

## 6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Tutorial: Network Reliability

Previously, we learned the concept of independent experiments. In this exercise, we'll see how the seemingly simple idea of independence can help us understand the behavior of quite complex systems. In particular, we'll combine the concept of independence with the idea of divide and conquer, where we break a larger system into smaller components, and then using independent properties to glue them back together.

Now, let's take a look at the problem. We are given a network of connected components, and each component can be good with probability  $P$  or bad otherwise. All components are independent from each other. We say the system is operational if there exists a path connecting point A here to point B that goes through only the good components. And we'd like to understand, what is the probability that the system is operational? Which we'll denote by  $P$  of A to B.

Although the problem might seem a little complicated at the beginning, it turns out only two structures really matter. So let's look at each of them. In the first structure, which we call the serial structure, we have a collection of  $k$  components, each one having probability  $P$  being good, connected one next to each other in a serial line.

Now, in this structure, in order for there to be a good path from A to B, every single one of the components must be working. So the probability of having a good path from A to B is simply  $P$  times  $P$ , so on and so, repeated  $k$  times, which is  $P$  raised to the  $k$  power. Know that the reason we can write the probability this way, in terms of this product, is because of the independence property.

Now, the second useful structure is parallel structure. Here again, we have  $k$  components one, two, through  $k$ , but this time they're connected in parallel to each other, namely they start from one point here and end at another point here, and this holds for every single component. Now, for the parallel structure to work, namely for there to exist a good path from A to B, it's easy to see that as long as one of these components works the whole thing will work. So the probability of A to B is the probability that at least one of these components works. Or in other words, the probability of the complement of the event where all components fail.

Now, if each component has probability  $P$  to be good, then the probability that all key components fail is  $1$  minus  $P$  raised to the  $k$ th power. Again, having this expression means that we have used the property of independence, and that is the probability of having a good parallel structure. Now, there's one more observation that will be useful for us. Just like how we define two components to be independent, we can also find two collections of components to be independent from each other.

For example, in this diagram, if we call the components between points C and E as collection two, and the components between E and B as collection three. Now, if we assume that each component in both collections-- they're completely independent from each other, then it's not

hard to see that collection two and three behave independently. And this will be very helpful in getting us the breakdown from complex networks to simpler elements.

Now, let's go back to the original problem of calculating the probability of having a good path from point big A to point big B in this diagram. Based on that argument of independent collections, we can first divide the whole network into three collections, as you see here, from A to C, C to E and E to B. Now, because they're independent and in a serial structure, as seen by the definition of a serial structure here, we see that the probability of A to B can be written as a probability of A to C multiplied by C to E, and finally, E to B.

Now, the probability of A to C is simply  $P$  because the collection contains only one element. And similarly, the probability of E to B is not that hard knowing the parallel structure here. We see that collection three has two components in parallel, so this probability will be given by  $1 - (1 - P)^2$ . And it remains to calculate just the probability of having a good path from point C to point E. To get a value for  $P_{C \text{ to } E}$ , we notice again, that this area can be treated as two components,  $C_1$  to E and  $C_2$  to E connected in parallel.

And using the parallel law we get this probability is  $1 - (1 - P_{C_1 \text{ to } E})(1 - P_{C_2 \text{ to } E})$ . Know that I'm using two different characters,  $C_1$  and  $C_2$ , to denote the same node, which is C. This is simply for making it easier to analyze two branches where they actually do note the same node. Now  $P_{C_1 \text{ to } E}$  is another serial connection of these three elements here with another component. So the first three elements are connected in parallel, and we know the probability of that being successful is  $1 - (1 - P)^3$ , and the last one is  $P$ .

And finally,  $P_{C_2 \text{ to } E}$ . It's just a single element component with probability of successful being  $P$ . At this point, there is no longer any unknown variables, and we have indeed obtained exact values for all the quantities that we're interested in. So starting from this equation, we can plug in the values for  $P_{C_2 \text{ to } E}$ ,  $P_{C_1 \text{ to } E}$  back here, and then further plug in  $P_{C \text{ to } E}$  back here. That will give us the final solution, which is given by the following somewhat complicated formula.

So in summary, in this problem, we learned how to use the independence property among different components to break down the entire fairly complex network into simple modular components, and use the law of serial and parallel connections to put the probabilities back together in common with the overall success probability of finding a path from A to B.

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