Lecture 6

- **Readings:** Sections 2.4-2.6

**Lecture outline**
- Review: PMF, expectation, variance
- Conditional PMF
- Geometric PMF
- Total expectation theorem
- Joint PMF of two random variables

**Review**
- Random variable \( X \): function from sample space to the real numbers
- PMF (for discrete random variables):
  \[ p_X(x) = P(X = x) \]
- Expectation:
  \[ E[X] = \sum_x x p_X(x) \]
  \[ E[g(X)] = \sum_x g(x) p_X(x) \]
  \[ E[\alpha X + \beta] = \alpha E[X] + \beta \]

\[ E\left[ X - E[X] \right] = \]

\[ \operatorname{var}(X) = \mathbb{E}(X - \mathbb{E}[X])^2 = \sum_x (x - \mathbb{E}[X])^2 p_X(x) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \]

**Standard deviation:** \( \sigma_X = \sqrt{\operatorname{var}(X)} \)

**Random speed**
- Traverse a 200 mile distance at constant but random speed \( V \)
  
  \[
  p_V(v) = \begin{cases} 
  1/2 & \text{if } v = 1 \\
  1/2 & \text{if } v = 200 \\
  0 & \text{otherwise}
  \end{cases}
  \]

  - \( d = 200, \ T = t(V) = 200/V \)
  - \( E[V] = \)
  - \( \operatorname{var}(V) = \)
  - \( \sigma_V = \)

**Average speed vs. average time**
- Traverse a 200 mile distance at constant but random speed \( V \)

  \[
  p_V(v) = \begin{cases} 
  1/2 & \text{if } v = 1 \\
  1/2 & \text{if } v = 200 \\
  0 & \text{otherwise}
  \end{cases}
  \]

  - time in hours = \( T = t(V) = \)
  - \( E[T] = \mathbb{E}[t(V)] = \sum_v t(v) p_V(v) = \)
  - \( E[T V] = 200 \neq E[T] \cdot E[V] \)
  - \( E[200/V] = E[T] \neq 200/E[V]. \)
### Conditional PMF and expectation

- \( p_{X \mid A}(x) = P(X = x \mid A) \)
- \( E[X \mid A] = \sum_x x p_{X \mid A}(x) \)

### Geometric PMF

- \( X \): number of independent coin tosses until first head
  \[ p_X(k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots \]
  \[ E[X] = \sum_{k=1}^{\infty} kp_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p \]
- Memoryless property: Given that \( X > y \), the r.v. \( X - 2 \) has same geometric PMF

### Total Expectation theorem

- Partition of sample space into disjoint events \( A_1, A_2, \ldots, A_n \)

\[ P(B) = P(A_1)P(B \mid A_1) + \cdots + P(A_n)P(B \mid A_n) \]
\[ p_X(x) = P(A_1)p_{X \mid A_1}(x) + \cdots + P(A_n)p_{X \mid A_n}(x) \]
\[ E[X] = P(A_1)E[X \mid A_1] + \cdots + P(A_n)E[X \mid A_n] \]

- Geometric example:
  \( A_1: \{X = 1\}, \quad A_2: \{X > 1\} \)
  \[ E[X] = P(X = 1)E[X \mid X = 1] + P(X > 1)E[X \mid X > 1] \]
- Solve to get \( E[X] = 1/p \)

### Joint PMFs

- \( p_{X,Y}(x, y) = P(X = x \text{ and } Y = y) \)
- \( p_X(x) = \sum_y p_{X,Y}(x, y) \)
- \( p_{X \mid Y}(x \mid y) = \frac{P(X = x \mid Y = y)}{P_Y(y)} = \frac{p_{X,Y}(x, y)}{p_Y(y)} \)
- \( \sum_x p_{X \mid Y}(x \mid y) = 1 \)