Recitation 7 Solutions  
September 30, 2010

1. See the textbook, Problem 2.35, page 130.

2. (a) 

\[ p_X(1) = P(X = 1, Y = 1) + P(X = 1, Y = 2) + P(X = 1, Y = 3) = \frac{1}{12} + \frac{2}{12} + \frac{1}{12} = \frac{1}{3} \]

(b) The solution is a sketch of the following conditional PMF:

\[
 p_{Y|X}(y \mid 1) = \frac{p_{Y,X}(y,1)}{p_X(1)} = \begin{cases} 
 1/4, & \text{if } y = 1, \\
 1/2, & \text{if } y = 2, \\
 1/4, & \text{if } y = 3, \\
 0, & \text{otherwise}. 
\end{cases}
\]

(c) \[ E[Y \mid X = 1] = \sum_{y=1}^{3} y p_{Y|X}(y \mid 1) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2 \]

(d) Assume that X and Y are independent. Because \( p_{X,Y}(3,1) = 0 \) and \( p_Y(1) = 1/4, p_X(3) \) must equal zero. This further implies \( p_{X,Y}(3,2) = 0 \) and \( p_{X,Y}(3,3) = 0 \). All the remaining probability mass must go to \((X, Y) = (2, 2)\), making \( p_{X,Y}(2,2) = 5/12, p_X(2) = 8/12, \) and \( p_Y(2) = 7/12 \). However, \( p_{X,Y}(2,2) \neq p_X(2) \cdot p_Y(2) \), contradicting the assumption; thus X and Y are not independent.

A simpler explanation uses only two X values and two Y values for which all four \((X, Y)\) pairs have specified probabilities. Note that if X and Y are independent, then \( p_{X,Y}(1,3)/p_{X,Y}(1,1) \) and \( p_{X,Y}(2,3)/p_{X,Y}(2,1) \) must be equal because they must both equal \( p_Y(3)/p_Y(1) \). This necessary equality does not hold, so X and Y are not independent.

(e) Knowing that X and Y are conditionally independent given B, we must have

\[ \frac{p_{X,Y}(1,1)}{p_X(1) p_X(1)} = \frac{p_{X,Y}(2,1)}{p_X(2) p_X(2)} \]

since the \((X, Y)\) pairs in the equality are all in B. Thus

\[ p_{X,Y}(2,2) = \frac{p_{X,Y}(1,2)p_{X,Y}(2,1)}{p_{X,Y}(1,1)} = \frac{(2/12)(2/12)}{1/12} = \frac{4}{12} = \frac{1}{3}. \]

(f) Since \( P(B) = 9/12 = 3/4 \), we normalize to obtain \( p_{X,Y|B}(2,2) = \frac{p_{X,Y}(2,2)}{P(B)} = 4/9. \)

3. See the textbook, Problem 2.33, page 128.