1. Let $X$ be a discrete random variable that takes the values 1 with probability $p$ and $-1$ with probability $1-p$. Let $Y$ be a continuous random variable independent of $X$ with the Laplacian (two-sided exponential) distribution

$$f_Y(y) = \frac{1}{2} \lambda e^{-\lambda|y|},$$

and let $Z = X + Y$. Find $P(X = 1 \mid Z = z)$. Check that the expression obtained makes sense for $p \to 0^+$, $p \to 1^-$, $\lambda \to 0^+$, and $\lambda \to \infty$.

2. Let $Q$ be a continuous random variable with PDF

$$f_Q(q) = \begin{cases} 6q(1-q), & \text{if } 0 \leq q \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

This $Q$ represents the probability of success of a Bernoulli random variable $X$, i.e.,

$$P(X = 1 \mid Q = q) = q.$$

Find $f_{Q|X}(q|x)$ for $x \in \{0, 1\}$ and all $q$.

3. Let $X$ have the normal distribution with mean 0 and variance 1, i.e.,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Also, let $Y = g(X)$ where

$$g(t) = \begin{cases} -t, & \text{for } t \leq 0; \\ \sqrt{t}, & \text{for } t > 0, \end{cases}$$

as shown to the right.

Find the probability density function of $Y$. 

$$-5 \quad 0 \quad 5$$

$$\theta(t)$$

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$$