Recitation 11 Solutions  
October 14, 2010  

1. We need to apply the version of Bayes rule for a discrete random variable conditioned on a continuous random variable:

\[ p_{X|Z}(x \mid z) = \frac{p_X(x) f_{Z|X}(z \mid x)}{f_Z(z)} = \frac{p_X(x) f_{Z|X}(z \mid x)}{\sum_{k=0}^{\infty} p_X(k) f_{Z|X}(z \mid k)}. \]

Specifically,

\[ P(X = 1 \mid Z = z) = p_{X|Z}(1 \mid z) = \frac{p_X(1) f_{Z|X}(z \mid 1)}{\sum_{k=0}^{\infty} p_X(k) f_{Z|X}(z \mid k)} = \frac{p_1 e^{-\lambda z |z - 1|}}{(1-p) \frac{1}{2} \lambda e^{-\lambda z |z + 1|} + p \frac{1}{2} \lambda e^{-\lambda z |z - 1|}}. \]

The final manipulations are to ease interpretations for \( p \to 0^+ \), \( p \to 1^- \), \( \lambda \to 0^+ \), and \( \lambda \to \infty \). Easily

\[ \lim_{p \to 0^+} P(X = 1 \mid Z = z) = 0 \quad \text{and} \quad \lim_{p \to 1^-} P(X = 1 \mid Z = z) = 1; \]

these make sense because the observation \( z \) should become unimportant when value of \( X \) becomes certain without it. Next,

\[ \lim_{\lambda \to 0^+} P(X = 1 \mid Z = z) = p, \]

which makes sense because the distribution of \( Y \) becomes very flat as \( \lambda \to 0^+ \), making the observation uninformative. Finally,

\[ \lim_{\lambda \to \infty} P(X = 1 \mid Z = z) = \begin{cases} 1, & \text{if } |z + 1| > |z - 1|, \\ 0, & \text{if } |z + 1| < |z - 1|, \end{cases} \]

this makes sense because \( \lambda \to \infty \) makes the \( Y \) negligible.

2. We need to apply the version of Bayes rule for a continuous random variable conditioned on a discrete random variable:

\[ f_{Q|X}(q \mid x) = \frac{f_Q(q) p_{X|Q}(x \mid q)}{p_X(x)} = \frac{f_Q(q) p_{X|Q}(x \mid q)}{\int_0^1 f_Q(q) p_{X|Q}(x \mid q) dq}. \]

For \( x = 0 \) and \( q \in [0, 1] \),

\[ f_{Q|X}(q \mid 0) = \frac{f_Q(q) p_{X|Q}(0 \mid q)}{\int_0^1 f_Q(q) p_{X|Q}(0 \mid q) dq} = \frac{6q(1 - q) \cdot (1 - q)}{1/2} = 12q(1 - q)^2. \]
For $x = 1$ and $q \in [0,1]$,

$$f_{Q|X}(q \mid 1) = \frac{f_Q(q)p_{X|Q}(1 \mid q)}{\int_0^1 f_Q(q)p_{X|Q}(1 \mid q)\,dq} = \frac{6q(1 - q)\cdot q}{\int_0^1 6q(1 - q)q\,dq} = \frac{6q(1 - q)\cdot q}{1/2} = 12q^2(1 - q).$$

The distributions $f_Q(q)$, $f_{Q|X}(q \mid 0)$, and $f_{Q|X}(q \mid 1)$ are all in the family of beta distributions, which arise again in Chapter 8.

3. Because of the definition of $g$, the random variable $Y$ takes on only nonnegative values. Thus $f_Y(y) = 0$ for any negative $y$. For $y > 0$,

$$F_Y(y) = P(Y \leq y) = P(X \in [-y,0]) + P(X \in (0,y^2]) = (F_X(0) - F_X(-y)) + (F_X(y^2) - F_X(0)) = F_X(y^2) - F_X(-y).$$

Taking the derivative of $F_Y(y)$ (and using the chain rule),

$$f_Y(y) = 2yf_X(y^2) + f_X(-y) = \frac{1}{\sqrt{2\pi}} \left( 2ye^{-y^4/2} + e^{-y^2/2} \right).$$