Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Tutorial 5: Solutions

1. (a) Let A be the event that the machine is functional. Conditioned on the random variable Q taking on a particular value q, $\mathbf{P}(A|Q=q)=q$. Using the continuous form of the total probability theorem, the probability of event A is given by:

$$\begin{aligned}\mathbf{P}(A) &= \int_0^1 \mathbf{P}(A|Q=q)f_Q(q)dq \\ &= \int_0^1 q\,dq \\ &= 1/2\end{aligned}$$

(b) Let B be the event that the machine is functional on m out of the last n days. Conditioned on random variable Q taking on value q (a probability q of being functional) the probability of event B is binomial with n trials, m successes, and a probability q of success in each trial. Again using the total probability theorem, the probability of event B is given by:

$$\begin{aligned}\mathbf{P}(B) &= \int_0^1 \mathbf{P}(B|Q=q)f_Q(q)dq \\ &= \int_0^1 \binom{n}{m} q^m(1-q)^{n-m}f_Q(q)dq \\ &= \binom{n}{m}\frac{m!(n-m)!}{(n+1)!}\end{aligned}$$

We then find the distribution on Q conditioned on event B using Bayes rule:

$$\begin{aligned}f_{Q|B}(q) &= \frac{\mathbf{P}(B|Q=q)f_Q(q)}{\mathbf{P}(B)} \\ &= \frac{q^m(1-q)^{n-m}}{\frac{m!(n-m)!}{(n+1)!}} \quad 0\le q\le 1, \quad n\ge m.\end{aligned}$$

2. Since $Y=|X|$ you can visualize the PDF for any given y as

$$f_Y(y) = \begin{cases} f_X(y)+f_X(-y), & \text{if } y\ge 0, \\ 0, & \text{if } y<0.\end{cases}$$

Also note that since $Y=|X|$, $Y\ge 0$.

(a) $f_X(x)=\begin{cases}\frac{1}{3}, & \text{if } -2<x\le 1, \\ 0, & \text{otherwise.}\end{cases}$

So, $f_X(x)$ for $-1\le x\le 0$ gets added to $f_X(x)$ for $0\le x\le 1$:

$$f_Y(y) = \begin{cases} 2/3, & \text{if } 0\le y\le 1, \\ 1/3, & \text{if } 1<y\le 2, \\ 0, & \text{otherwise.}\end{cases}$$

(b) Here we are told $X > 0$. So there are no negative values of X that need to be considered. Thus,

$$f_Y(y) = f_X(y) = \begin{cases} 2e^{-2y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

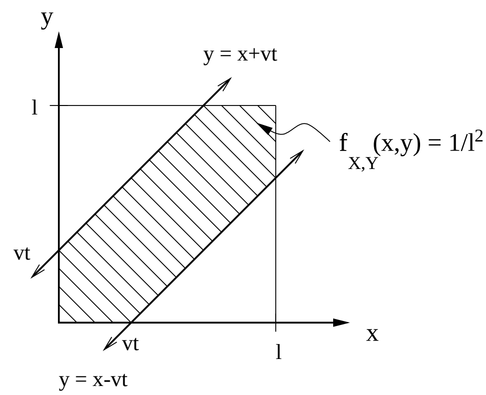
(c) As explained in the beginning, $f_Y(y) = f_X(y) + f_X(-y)$.

3. We want to compute the CDF of the ambulance's travel time T, $\mathbf{P}(T \leq t) = \mathbf{P}(|X - Y| \leq vt)$, where X and Y are the locations of the ambulance and accident (uniform over $[0, l]$). Since X and Y are independent, we know:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{l^2} & , \text{ if } 0 \leq x, y \leq l \\ 0 & , \text{ otherwise} \end{cases} .$$

$$\begin{aligned} \mathbf{P}(T \leq t) &= \mathbf{P}(|X - Y| \leq vt) = \mathbf{P}(-vt \leq Y - X \leq vt) \\ &= \mathbf{P}(X - vt \leq Y \leq X + vt) \end{aligned}$$

We can see that $\mathbf{P}(X - vt \leq Y \leq X + vt)$ corresponds to the integral of the joint density of X and Y over the shaded region in the figure below:



Therefore, because the joint density is uniform over the entire region, we have:

$$F_T(t) = (1/l^2) \times (\text{Shaded area}) = \begin{cases} 0 & , \text{ if } t < 0 \\ \frac{2vt}{l} - \frac{(vt)^2}{l^2} & , \text{ if } 0 \leq t < \frac{l}{v} \\ 1 & , \text{ if } t \geq \frac{l}{v} \end{cases} .$$

By differentiating the CDF, we find the density of T:

$$f_T(t) = \begin{cases} \frac{2v}{l} - \frac{2v^2 t}{l^2} & , \text{ if } 0 \leq t \leq \frac{l}{v} \\ 0 & , \text{ otherwise} \end{cases} .$$

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