LECTURE 14

The Poisson process

- **Readings:** Start Section 6.2.

Lecture outline

- Review of Bernoulli process
- Definition of Poisson process
- Distribution of number of arrivals
- Distribution of interarrival times
- Other properties of the Poisson process

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Definition of the Poisson process

\[ P(k, \tau) = \text{Prob. of } k \text{ arrivals in interval of duration } \tau \]

- **Time homogeneity:**
  \[ P(k, \delta) = P(k, \tau) \]
  Numbers of arrivals in disjoint time intervals are independent

- **Small interval probabilities:**
  For VERY small \( \delta \):
  \[
P(k, \delta) \approx \begin{cases} 
1 - \lambda \delta, & \text{if } k = 0; \\
\lambda \delta, & \text{if } k = 1; \\
0, & \text{if } k > 1.
\end{cases}
\]
  \( \lambda \): “arrival rate”

PMF of Number of Arrivals \( N \)

- Finely discretize \([0, t]\): approximately Bernoulli
- \( N_t \) (of discrete approximation): binomial
- Taking \( \delta \to 0 \) (or \( n \to \infty \)) gives:
  \[
P(k, \tau) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!}, \quad k = 0, 1, \ldots
\]
- \( E[N_t] = \lambda t, \quad \text{var}(N_t) = \lambda t \)
Example

- You get email according to a Poisson process at a rate of $\lambda = 5$ messages per hour. You check your email every thirty minutes.

- $\text{Prob}(\text{no new messages}) =$

- $\text{Prob}(\text{one new message}) =$

Interarrival Times

- $Y_k$ time of $k$th arrival

- Erlang distribution:
  
  $$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0$$

  - Memoryless property: The time to the next arrival is independent of the past

Bernoulli/Poisson Relation

<table>
<thead>
<tr>
<th></th>
<th>POISSON</th>
<th>BERNOULLI</th>
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</thead>
<tbody>
<tr>
<td>Times of Arrival</td>
<td>Continuous</td>
<td>Discrete</td>
</tr>
<tr>
<td>Arrival Rate</td>
<td>$\lambda$/unit time</td>
<td>$p$/per trial</td>
</tr>
<tr>
<td>PMF of # of Arrivals</td>
<td>Poisson</td>
<td>Binomial</td>
</tr>
<tr>
<td>Interarrival Time Distr.</td>
<td>Exponential</td>
<td>Geometric</td>
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<tr>
<td>Time to $k$th arrival</td>
<td>Erlang</td>
<td>Pascal</td>
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</tbody>
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Merging Poisson Processes

- Sum of independent Poisson random variables is Poisson

- Merging of independent Poisson processes is Poisson

  - Red bulb flashes (Poisson)
  
  - Green bulb flashes (Poisson)

  - What is the probability that the next arrival comes from the first process?
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