LECTURE 15

Poisson process — II

- Readings: Finish Section 6.2.
- Review of Poisson process
- Merging and splitting
- Examples
- Random incidence

Review
- Defining characteristics
  - Time homogeneity: \( P(k, \tau) \)
  - Independence
  - Small interval probabilities (small \( \delta \)):
    \[
    P(k, \delta) \approx \begin{cases} 
      1 - \lambda \delta, & \text{if } k = 0, \\
      \lambda \delta, & \text{if } k = 1, \\
      0, & \text{if } k > 1.
    \end{cases}
    \]
- \( N_\tau \) is a Poisson r.v., with parameter \( \lambda \tau \):
  \[
  P(k) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!}, \quad k = 0, 1, \ldots
  \]
  \[ \mathbb{E}[N_\tau] = \text{var}(N_\tau) = \lambda \tau \]
- Interarrival times (\( k = 1 \)): exponential:
  \[
  f_{T_1}(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \quad \mathbb{E}[T_1] = 1/\lambda
  \]
- Time \( Y_k \) to \( k \)th arrival: Erlang(\( k \)):
  \[
  f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0
  \]

Poisson fishing
- Assume: Poisson, \( \lambda = 0.6 \)/hour.
  - Fish for two hours.
  - if no catch, continue until first catch.
  
  a) \( P(\text{fish for more than two hours}) = \)
  
  b) \( P(\text{fish for more than two and less than five hours}) = \)
  
  c) \( P(\text{catch at least two fish}) = \)
  
  d) \( \mathbb{E}[\text{number of fish}] = \)
  
  e) \( \mathbb{E}[\text{future fishing time} | \text{fished for four hours}] = \)
  
  f) \( \mathbb{E}[\text{total fishing time}] = \)

Merging Poisson Processes (again)
- Merging of independent Poisson processes is Poisson

\[
\begin{array}{cccc}
\lambda_1 & & & \\
\lambda_2 & & & \\
\text{Red bulb flashes (Poisson)} & & & \text{All flashes (Poisson)} \\
\text{Green bulb flashes (Poisson)} & & & \\
- & What is the probability that the next arrival comes from the first process?
\end{array}
\]
**Light bulb example**

- Each light bulb has independent, exponential(λ) lifetime.
- Install three light bulbs. Find expected time until last light bulb dies out.

**Splitting of Poisson processes**

- Assume that email traffic through a server is a Poisson process. Destinations of different messages are independent.

```
Email Traffic leaving MIT
\lambda
\rightarrow
\begin{array}{c}
USA \\
(1 - p)\lambda
\end{array}
\rightarrow
\begin{array}{c}
Foreign \\
p\lambda
\end{array}
```

- Each output stream is Poisson.

**Random incidence for Poisson**

- Poisson process that has been running forever.
- Show up at some “random time” (really means “arbitrary time”)
- What is the distribution of the length of the chosen interarrival interval?

**Random incidence in “renewal processes”**

- Series of successive arrivals
  - i.i.d. interarrival times
    - but not necessarily exponential
- Example:
  - Bus interarrival times are equally likely to be 5 or 10 minutes
  - If you arrive at a “random time”:
    - what is the probability that you selected a 5 minute interarrival interval?
    - what is the expected time to next arrival?