LECTURE 17

Markov Processes – II

- **Readings:** Section 7.3

Lecture outline

- Review

- Steady-State behavior
  - Steady-state convergence theorem
  - Balance equations

- Birth-death processes

Review

- Discrete state, discrete time, time-homogeneous
  - Transition probabilities $p_{ij}$
  - Markov property

- $r_{ij}(n) = P(X_n = j \mid X_0 = i)$

- Key recursion:
  $$r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$$

Warmup

Recurrence graph: $P(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1)$ =

$P(X_4 = 7 \mid X_0 = 2)$ =

**Recurrent and transient states**

- State $i$ is **recurrent** if:
  - starting from $i$,
  - and from wherever you can go,
  - there is a way of returning to $i$

- If not recurrent, called **transient**

**Recurrent class:**
- collection of recurrent states that “communicate” to each other and to no other state

Periodic states

- The states in a recurrent class are **periodic** if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group
Steady-State Probabilities

- Do the $r_{ij}(n)$ converge to some $\pi_j$? (independent of the initial state $i$)
- Yes, if:
  - recurrent states are all in a single class, and
  - single recurrent class is not periodic
- Assuming “yes,” start from key recursion
  \[ r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj} \]
  - take the limit as $n \to \infty$
  \[ \pi_j = \sum_k \pi_k p_{kj}, \quad \text{for all } j \]
  - Additional equation:
    \[ \sum_j \pi_j = 1 \]

Visit frequency interpretation

- (Long run) frequency of being in $j$: $\pi_j$
- Frequency of transitions $k \to j$: $\pi_k p_{kj}$
- Frequency of transitions into $j$: $\sum_k \pi_k p_{kj}$

Example Birth-death processes

- Special case: $p_i = p$ and $q_i = q$ for all $i$
  - $\rho = p/q$ = load factor
    \[ \pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho \]
    \[ \pi_i = \pi_0 \rho^i, \quad i = 0, 1, \ldots, m \]
- Assume $p < q$ and $m \approx \infty$
  \[ \pi_0 = 1 - \rho \]
  \[ E[X_0] = \frac{\rho}{1 - \rho} \quad \text{(in steady-state)} \]