LECTURE 19
Limit theorems – I

• Readings: Sections 5.1-5.3; start Section 5.4

• $X_1, \ldots, X_n$ i.i.d.
  $$M_n = \frac{X_1 + \cdots + X_n}{n}$$
  What happens as $n \to \infty$?

• Why bother?
• A tool: Chebyshev’s inequality
• Convergence “in probability”
• Convergence of $M_n$
  (weak law of large numbers)

Chebyshev’s inequality

• Random variable $X$
  (with finite mean $\mu$ and variance $\sigma^2$)
  $$\sigma^2 = \int (x - \mu)^2 f_X(x) \, dx$$
  $$\geq \int_{-\infty}^{-c} (x - \mu)^2 f_X(x) \, dx + \int_{c}^{\infty} (x - \mu)^2 f_X(x) \, dx$$
  $$\geq c^2 \cdot P(|X - \mu| \geq c)$$

  $$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

Convergence “in probability”

• Sequence of random variables $Y_n$

• converges in probability to a number $a$:
  "(almost all) of the PMF/PDF of $Y_n$,
  eventually gets concentrated
  (arbitrarily) close to $a"

• For every $\epsilon > 0$,
  $$\lim_{n \to \infty} P(|Y_n - a| \geq \epsilon) = 0$$

Deterministic limits

• Sequence $a_n$
  Number $a$

• $a_n$ converges to $a$
  $$\lim_{n \to \infty} a_n = a$$
  “$a_n$ eventually gets and stays
  (arbitrarily) close to $a$”

• For every $\epsilon > 0$, there exists $n_0$, such that for every $n \geq n_0$, we have $|a_n - a| \leq \epsilon$. 

Convergence “in probability”

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  eventually gets concentrated
  (arbitrarily) close to $a"

• For every $\epsilon > 0$,
  $$\lim_{n \to \infty} P(|Y_n - a| \geq \epsilon) = 0$$

Does $Y_n$ converge?
Convergence of the sample mean  
(Weak law of large numbers)

• \( X_1, X_2, \ldots \) i.i.d.  
  finite mean \( \mu \) and variance \( \sigma^2 \)  
  \[
  M_n = \frac{X_1 + \cdots + X_n}{n}
  \]

• \( E[M_n] = \)

• \( \text{Var}(M_n) = \)

\[
\Pr(\lvert M_n - \mu \rvert \geq \epsilon) \leq \frac{\text{Var}(M_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}
\]

• \( M_n \) converges in probability to \( \mu \)

The pollster's problem

• \( f \): fraction of population that “…”

• \( i \)th (randomly selected) person polled:
  \[
  X_i = \begin{cases} 
  1, & \text{if yes,} \\
  0, & \text{if no.} 
  \end{cases}
  \]

• \( M_n = (X_1 + \cdots + X_n)/n \)
  fraction of “yes” in our sample

• Goal: 95% confidence of \( \leq 1\% \) error
  \[
  \Pr(|M_n - f| \geq 0.01) \leq 0.05
  \]

• Use Chebyshev’s inequality:
  \[
  \Pr(|M_n - f| \geq 0.01) \leq \frac{\sigma^2_{M_n}}{(0.01)^2} = \frac{\sigma^2}{n(0.01)^2} \leq \frac{1}{4n(0.01)^2}
  \]

• If \( n = 50,000 \),  
  then \( \Pr(|M_n - f| \geq 0.01) \leq 0.05 \)  
  (conservative)

Different scalings of \( M_n \)

• \( X_1, \ldots, X_n \) i.i.d.  
  finite variance \( \sigma^2 \)

• Look at three variants of their sum:

• \( S_n = X_1 + \cdots + X_n \)  
  variance \( n\sigma^2 \)

• \( M_n = \frac{S_n}{n} \)  
  variance \( \sigma^2/n \)  
  converges “in probability” to \( E[X] \) (WLLN)

• \( \frac{S_n}{\sqrt{n}} \)  
  constant variance \( \sigma^2 \)

  – Asymptotic shape?

The central limit theorem

• “Standardized” \( S_n = X_1 + \cdots + X_n \):
  \[
  Z_n = \frac{S_n - E[S_n]}{\sigma S_n} = \frac{S_n - nE[X]}{\sqrt{n}\sigma}
  \]

  – zero mean
  – unit variance

• Let \( Z \) be a standard normal r.v.  
  (zero mean, unit variance)

• \textbf{Theorem:} For every \( c \):
  \[
  \Pr(Z_n \leq c) \rightarrow \Pr(Z \leq c)
  \]

• \( \Pr(Z \leq c) \) is the standard normal CDF,  
  \( \Phi(c) \), available from the normal tables