Problem Set 5

Due: October 21

Reading: Course notes on number theory.

Problem 1. Suppose that one domino can cover exactly two squares on a chessboard, either vertically or horizontally.

(a) Can you tile an $8 \times 8$ chessboard with 32 dominos?

(b) Can you tile an $8 \times 8$ chessboard with 31 dominos if opposite corners are removed?

(c) Now suppose that an assortment of squares are removed from a chessboard. An example is shown below.
Given a truncated chessboard, show how to construct a bipartite graph $G$ that has a perfect matching if and only if the chessboard can be tiled with dominos.

(d) Based on this construction and Hall’s theorem, can you state a necessary and sufficient condition for a truncated chessboard to be tilable with dominos? Try not to mention graphs or matchings!

**Problem 2.** Prove that $\gcd(ka, kb) = k \cdot \gcd(a, b)$ for all $k > 0$.

**Problem 3.** Suppose that $a \equiv b \pmod{n}$ and $n > 0$. Prove or disprove the following assertions:

(a) $a^c \equiv b^c \pmod{n}$ where $c \geq 0$

(b) $a^a \equiv b^b \pmod{n}$ where $a, b, \geq 0$

**Problem 4.** An inverse of $k$ modulo $n > 1$ is an integer, $k^{-1}$, such that

$$k \cdot k^{-1} \equiv 1 \pmod{n}.$$ 

Show that $k$ has an inverse iff $\gcd(k, n) = 1$. *Hint: We saw how to prove the above when $n$ is prime.*

**Problem 5.** Here is a long run of composite numbers:

$$114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126$$

Prove that there exist arbitrarily long runs of composite numbers. Consider numbers a little bigger than $n!$ where $n! = n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1$. 
Problem 6. Take a big number, such as 37273761261. Sum the digits, where every other one is negated:

\[ 3 + (−7) + 2 + (−7) + 3 + (−7) + 6 + (−1) + 2 + (−6) + 1 = −11 \]

As it turns out, the original number is a multiple of 11 if and only if this sum is a multiple of 11.

(a) Use a result from elsewhere on this problem set to show that \( 10^k \equiv −1^k \pmod{11} \).

(b) Using this fact, explain why the procedure above works.

Problem 7. Let \( S_k = 1^k + 2^k + \ldots + (p - 1)^k \), where \( p \) is an odd prime and \( k \) is a positive multiple of \( p - 1 \). Use Fermat’s theorem to prove that \( S_k \equiv -1 \pmod{p} \).
Student’s Solutions to Problem Set 5

Your name:

Due date: October 21

Submission date:

Circle your TA: David Jelani Sayan Hanson

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.

2. I collaborated on this assignment with:
   - got help from:
   - and referred to:

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1People other than course staff.

2Give citations to texts and material other than the Fall ’02 course materials.