

In-Class Problems Week 10, Wed.

Problem 1. Find the coefficients of

(a) x^5 in $(1 + x)^{11}$

(b) x^8y^9 in $(3x + 2y)^{17}$

(c) a^6b^6 in $(a^2 + b^3)^5$

Problem 2. According to the Multinomial theorem, $(w + x + y + z)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, r_3, r_4} w^{r_1} x^{r_2} y^{r_3} z^{r_4}.$$

How many terms are there in the sum?

Combinatorial proofs of identities

Recall the basic plan for a combinatorial proof of an identity $x = y$:

1. Define a set S .
2. Show that $|S| = x$ by counting one way.
3. Show that $|S| = y$ by counting another way.
4. Conclude that $x = y$.

Problem 3. You want to choose a team of m people from a pool of n people for your startup company, and from these m people you want to choose k to be the team managers. You took 6.042, so you know you can do this in

$$\binom{n}{m} \binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k} \binom{n-k}{m-k}.$$

Before doing the reasonable thing—dump on your CFO or Harvard Business School—you decide to check his answer against yours.

- (a) Start by giving an *algebraic proof* that your CFO's formula agrees with yours.
- (b) Now give a *combinatorial argument* proving this same fact.

Problem 4. Now give a combinatorial proof of the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=1}^n k \binom{n}{k}$$

Hint: Let S be the set of all length- n sequences of 0's, 1's and a single *.

Learning to count takes practice! The following problems offer some.

Problem 5. A pizza house is having a promotional sale. Their commercial reads:

We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's 22,369,621 different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to 22,369,621. Unfortunately, $(2^9)^3/3!$ is obviously not an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

Problem 6. (a) In how many different ways can Blockbuster arrange 64 copies of *13 conversations about one thing*, 96 copies of *L'Auberge Espagnole* and 1 copy of *Matrix Revolutions* on a shelf? What if they are to be arranged in 5 shelves?

(b) Set A has r elements and set B has n elements. How many functions are there from A to B ? How many of them are injective (one-to-one)? How many of them are bijective?

(c) Find the number of 5-card hands in which every suit appears at most twice.

(d) How many paths are there from point $(0, 0)$ to $(50, 50)$ if every step increments one coordinate and leaves the other unchanged? if there are impassable boulders sitting at points $(10, 10)$ and $(20, 20)$? *Hint:* Count the number of paths going through $(10, 10)$, the number through $(20, 20)$, and use Inclusion-Exclusion.