Solutions to In-Class Problems Week 11, Fri.

Problem 1. (a) Verify that
\[
\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n.
\]

*Hint:* Use the fact that if \( A(x) = \sum_{n=0}^{\infty} a_n x^n \), then
\[
a_n = \frac{A^{(n)}(0)}{n!},
\]
where \( A^{(n)} \) is the \( n \)th derivative of \( A \).

*Solution.*
\[
\frac{d}{dx} \left( \frac{1}{1-x} \right)^k = k \left( \frac{1}{1-x} \right)^{(k+1)},
\]
\[
\frac{d^2}{(dx)^2} \left( \frac{1}{1-x} \right)^k = k(k+1) \left( \frac{1}{1-x} \right)^{(k+2)}
\]
\[
\frac{d^3}{(dx)^3} \left( \frac{1}{1-x} \right)^k = (k+1)(k+2) \left( \frac{1}{1-x} \right)^{(k+3)}
\]
\[
\vdots
\]
\[
\frac{d^n}{(dx)^n} \left( \frac{1}{1-x} \right)^k = (k+n-1) \cdots (k+2)(k+1) \left( \frac{1}{1-x} \right)^{(k+n)}.
\]

Now suppose \( (1-x)^{-k} = A(x) \). Then by the hint, we have
\[
a_n = \frac{A^{(n)}(0)}{n!}
\]
\[
= \frac{(k+n-1) \cdots (k+2)(k+1)(1-0)^{(k+n)}}{n!} \cdot \frac{(n+k-1)!}{(k-1)!} \cdot \frac{1}{n!}
\]
\[
= \frac{(n+k-1)!}{(n)! (k-1)! n!}
\]
\[
= \binom{n+k+1}{n}
\]

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(b) Let $S(x) := \sum_{k=1}^{\infty} k^2 x^k$. Explain why $S(x)/(1-x)$ is the generating function for the sums of squares. That is, the coefficient of $x^n$ in the series for $S(x)/(1-x)$ is $\sum_{k=1}^{n} k^2$.

Solution. 

\[
\left(\sum_{n=0}^{\infty} a_n x^n\right) \left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k \cdot 1\right) x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k\right) x^n
\]  

by the convolution formula for the product of series. For $S(x)$, the coefficient of $x^k$ is $a_k = k^2$, and

\[
S(x)/(1-x) = S(x) \left(\sum_{n=0}^{\infty} x^n\right),
\]

so (1) implies that the coefficient of $x^n$ in $S(x)/(1-x)$ is the sum of the first $n$ squares.

(c) Use the fact that

\[
S(x) = \frac{x(1+x)}{(1-x)^3},
\]

and the previous part to prove that

\[
\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.
\]

Solution. We have

\[
\frac{S(x)}{1-x} = \frac{x(1+x)}{(1-x)^3} = \frac{x + x^2}{(1-x)^4},
\]

From part (a), the coefficient of $x^n$ in the series expansion of $1/(1-x)^4$ is

\[
\binom{n+3}{n} = \frac{(n+1)(n+2)(n+3)}{3!}.
\]

But by (2),

\[
\frac{S(x)}{1-x} = \frac{x}{(1-x)^4} + \frac{x^2}{(1-x)^4},
\]

so the coefficient of $x^n$ is the sum of the $(n-1)$st and $(n-2)$nd coefficients of $(1-x)^4$, namely,

\[
\frac{n(n+1)(n+2)}{3!} + \frac{(n-1)n(n+1)}{3!} = \frac{n(n+1)(2n+1)}{6}.
\]

(d) (Optional) How about a formula for the sum of cubes?

Solution. TBA
Problem 2. We are interested in generating functions for the number of different ways to compose a bag of \( n \) donuts subject to various restrictions. For each of the restrictions in (a)-(e) below, find a closed form for the corresponding generating function.

(a) All the donuts are chocolate and there are at least 3.
Solution.
\[
\frac{x^3}{1 - x}
\]

(b) All the donuts are glazed and there are at most 2.
Solution.
\[
1 + x + x^2
\]

(c) All the donuts are coconut and there are exactly 2 or there are none.
Solution.
\[
1 + x^2
\]

(d) All the donuts are plain and their number is a multiple of 4.
Solution.
\[
\frac{1}{1 - x^4} = \frac{1}{(1 - x)(1 + x)(1 + x^2)}
\]

(e) The donuts must be chocolate, glazed, coconut, or plain and:
• there must be at least 3 chocolate donuts, and
• there must be at most 2 glazed, and
• there must be exactly 0 or 2 coconut, and
• there must be a multiple of 4 plain.
Solution.
\[
\frac{x^3}{1 - x} (1 + x + x^2)(1 + x^2) \frac{1}{1 - x^4} = \frac{x^3(1 + x + x^2)(1 + x^2)}{(1 - x)(1 + x)(1 + x^2)} = (x^3 + x^4 + x^5) \frac{1}{(1 - x)^2(1 + x)}
\]
(f) Find a closed form for the number of ways to select \( n \) donuts subject to the constraints of the previous part.

**Solution.**

\[
\frac{1}{(1-x)^2(1+x)} = \frac{1/2}{(1-x)^2} + \frac{1/4}{1-x} + \frac{1/4}{1+x}
\]

so the \( n \)th coefficient in its generating function is

\[
\frac{n+1}{2} + \frac{1}{4} + \frac{(-1)^n}{4} = \frac{2n+3+(-1)^n}{4}
\]

The number ways to select \( n \) donuts is the sum of the \((n-3)\)rd, \((n-4)\)th, and \((n-5)\)th of these coefficients, namely

\[
\frac{2(n-3) + 2(n-4) + 2(n-5) + 9 + (-1)^{n-3} + (-1)^{n-4} + (-1)^{n-5}}{4} = \frac{6n-15 + (-1)^{n-1}}{4}
\]

\[\blacksquare\]

**Appendix**

**Products of Series**

Let

\[
A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad B(x) = \sum_{n=0}^{\infty} b_n x^n, \quad C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n.
\]

Then

\[c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0.\]