Solutions to In-Class Problems Week 15, Wed.

Gamblers Ruin

A gambler aims to gamble until he reaches a goal of $T$ dollars or until he runs out of money, in which case he is said to be “ruined.” He gambles by making a sequence of 1 dollar bets. If he wins an individual bet, his stake increases by one dollar. If he loses, his stake decreases by one dollar. In each bet, he wins with probability $p > 0$ and loses with probability $q := 1 - p > 0$. He is an overall winner if he reaches his goal and is an overall loser if he gets ruined.

In a fair game, $p = q = 1/2$. The gambler is more likely to win if $p > 1/2$ and less likely to win if $p < 1/2$.

With $T$ and $p$ fixed, the gambler’s probability of winning will depend on how much money he starts with. Let $w_n$ be the probability that he is a winner when his initial stake in $n$ dollars.

Problem 1. (a) What are $w_0$ and $w_T$?
Solution. $w_0 = 0$ and $w_T = 1$.

(b) Note that $w_n$ satisfies a linear recurrence
\[ w_{n+1} = aw_n + bw_{n-1} \] (1)
for some constants $a, b$ and $0 < n < T$. Write simple expressions for $a$ and $b$ in terms of $p$.
Solution. By Total Probability
\[ w_n = \Pr \{ \text{win game | win the first bet} \} \Pr \{ \text{win the first bet} \} + \]
\[ \Pr \{ \text{win game | lose the first bet} \} \Pr \{ \text{lose the first bet} \} \]
\[ = pw_{n+1} + q \Pr \{ w_{n-1} \}, \] (2)
so
\[ pw_{n+1} = w_n - qw_{n-1} \]
\[ w_{n+1} = \frac{w_n - qw_{n-1}}{p}. \] (3)
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So

\[ a = \frac{1}{p}, \quad b = -\frac{q}{p}. \]

**c** For \( n > T \), let \( w_n \) be defined by the recurrence (1), and let \( g(x) := \sum_{n=1}^{\infty} w_n x^n \) be the generating function for the sequence \( w_0, w_1, \ldots \). Verify that

\[ g(x) = \frac{w_1 x}{(1 - x)(1 - \frac{q}{p} x)}. \]  \hspace{1cm} (4)

**Solution.**

\[
\begin{align*}
g(x) &= w_0 + w_1 x + w_2 x^2 + w_3 x^3 + \cdots \\
xg(x)/p &= \frac{w_0 x}{p} + \frac{w_1 x^2}{p} + \frac{w_2 x^3}{p} + \cdots \\
(q/p) x^2 g(x) &= \frac{(q/p) w_0 x^2}{p} + \frac{(q/p) w_1 x^3}{p} + \cdots
\end{align*}
\]

so

\[
g(x) = \left( \frac{xg(x)}{p} - \frac{qx^2 g(x)}{p} \right) = w_0 + w_1 x - \frac{w_0 x}{p} = w_1 x,
\]

\[
g(x) \left( 1 - \frac{x}{p} + \frac{qx^2}{p} \right) = w_1 x. \]  \hspace{1cm} (5)

But

\[
1 - \frac{x}{p} + \frac{qx^2}{p} = (1 - x)\left(1 - \frac{q}{p} x\right) \]  \hspace{1cm} (6)

Combining (6) and (5) yields (4).

**d** Conclude that in an unfair game

\[ w_n = c + d \left( \frac{q}{p} \right)^n \]  \hspace{1cm} (7)

for some constants \( c, d \).

**Solution.** In an unfair game \( p/q \neq 1 \), so from (4), we know that there will be \( c, d \) such that

\[
g(x) = \frac{c}{1 - x} + \frac{d}{1 - \frac{q}{p} x} \]  \hspace{1cm} (8)

so \( w_n \) will be the corresponding combination of the coefficients of \( x^n \) in \( 1/(1 - x) \) and \( 1/(1 - (q/p)x) \), namely, (7).
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(e) Show that in an unfair game,

\[ w_n = \frac{(q/p)^n - 1}{(q/p)^T - 1}. \]

**Solution.** Given (4), we want \( c, d \) such that

\[ \frac{w_1 x}{(1-x)(1-\frac{q}{p}x)} = \frac{c}{1-x} + \frac{d}{1-\frac{q}{p}x}. \]

So \( c, d \) satisfy

\[ w_1 x = c(1-\frac{q}{p}x) + d(1-x). \]

Letting \( x = 1 \) gives

\[ c = \frac{w_1}{1-q/p}. \]

Letting \( x = p/q \) gives

\[ d = \frac{pw_1/q}{1-p/q} = \frac{w_1}{q/p-1} = -c. \]

So plugging into (7) gives

\[ w_n = \frac{w_1}{q/p-1} \left( \left( \frac{q}{p} \right)^n - 1 \right). \]  \hspace{1cm} (9)

Now we can solve for \( w_1 \), by letting \( n = T \) in (9):

\[ 1 = w_T = \frac{w_1}{q/p-1} \left( \left( \frac{q}{p} \right)^T - 1 \right) \]

so

\[ w_1 = \frac{(q/p - 1)}{(q/p)^T - 1}. \]

Combining this with (9) yields

\[ w_n = \frac{(q/p)^n - 1}{(q/p)^T - 1}. \]

\[ \blacksquare \]

(f) Verify that if \( 0 < a < b \), then

\[ \frac{a}{b} < \frac{a+1}{b+1}. \]

Conclude that if \( p < 1/2 \), then

\[ w_n < \left( \frac{p}{q} \right)^{T-n}. \]
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Solution.

\[
\frac{a}{b} = \frac{a(1 + 1/b)}{b(1 + 1/b)} = \frac{a + a/b}{b + 1} < \frac{a + 1}{b + 1}.
\]

So from the previous part, we have

\[
w_n = \frac{(q/p)^n - 1}{(q/p)^T - 1} < \frac{(q/p)^n}{(q/p)^T} = \left(\frac{q}{p}\right)^{n-T} = \left(\frac{p}{q}\right)^{T-n}.
\]

Problem 2. Show that in a fair game,

\[w_n = \frac{w}{T}.\]

Hint: Use equation (4) again.

Solution. This time \(p = q = 1/2\) so from (4),

\[g(x) = \frac{w_1x}{(1-x)^2}.\]

Now we need \(a, b\) such that

\[
\frac{w_1x}{(1-x)^2} = \frac{a}{1-x} + \frac{b}{(1-x)^2},
\]

so we will have

\[w_n = a + b(n + 1).\]

Solving for \(a, b\), we have from (10)

\[w_1x = a(1-x) + b.\]

Letting \(x = 0\) yields \(a = -b\) and \(x = 1\) yields \(b = w_1\), so

\[w_n = -w_1 + w_1(n + 1) = w_1n.\]

Also,

\[1 = w_T = w_1T\]

so

\[w_1 = \frac{1}{T},\]

and hence

\[w_n = \frac{n}{T}.\]
Problem 3. Now suppose $T = \infty$, that is, the gambler keeps playing until he is ruined. (Now there may be a positive probability that he actually plays forever.) Let $r$ be the probability that starting with $n > 0$ dollars, the gambler’s stake ever gets reduced to $n - 1$.

(a) Explain why

$$r = q + pr^2.$$

Solution. By Total Probability

$$r = \Pr \{\text{ever down $1$ | lose the first bet}\} \Pr \{\text{lose the first bet}\} +$$

$$\Pr \{\text{ever down $1$ | win the first bet}\} \Pr \{\text{win the first bet}\}$$

$$= q + p \Pr \{\text{ever down $1$ | win the first bet}\}$$

But

$$\Pr \{\text{ever down $1$ | win the first bet}\} = \Pr \{\text{ever down $2$}\}$$

$$= \Pr \{\text{being down the first $1$}\} \Pr \{\text{being down another $1$}\}$$

$$= r^2.$$

(b) Conclude that if $p \leq 1/2$, then $r = 1$.

Solution. $pr^2 - r + q$ has roots $q/p$ and 1. So $r = 1$ or $r = q/p$. But $1 \leq r$, which implies $r = 1$ when $q/p \geq 1$, that is, when $p \leq 1/2$.

In fact $r = q/p$ when $q/p < 1$, namely, when $p > 1/2$, but this requires an additional argument that we omit.

(c) Conclude that even in a fair game, the gambler is sure to get ruined no matter how much money he starts with!

Solution. The gambler gets ruined starting with initial stake $n = 1$ precisely if his initial stake goes down by 1 dollar, so his probability of ruin is $r$, which equals 1 in the fair case. The recurrence (1) will also hold in this $T = \infty$ case if we interpret $w_n$ as the probability of not being ruined, that is, the gambler wins if he can gamble forever. So $w_1$ is the probability he is not getting ruined starting with a 1 dollar stake, that is $w_1 = 1 - r = 0$. Since $w_0 = 0 = w_1$, the recurrence implies that $w_n = 0$ for all $n \geq 0$. 


(d) Let $t$ be the expected time for the gambler’s stake to go down by 1 dollar. Verify that

$$t = q + p(1 + 2t).$$

Conclude that starting with a 1 dollar stake in a fair game, the gambler can expect to play forever!

**Solution.** By Total Expectation

$$t = E [\text{#steps to be down $1 \mid lose the first bet}] \Pr \{\text{lose the first bet}\} +$$
$$E [\text{#steps to be down $1 \mid win the first bet}] \Pr \{\text{win the first bet}\}$$

$$= q + p E [1 + \text{#steps to be down $1 \mid win the first bet}] .$$

But

$$E [\text{#steps to be down $1 \mid win the first bet}]$$
$$= E [\text{#steps to be down $2}]$$
$$= E [\text{#steps to be down the first $1}] + E [\text{#steps to be down another $1}]$$
$$= 2t.$$

This implies the required formula $t = q + p(1 + 2t)$. If $p = 1/2$ we conclude that $t = 1 + t$, which means $t$ must be infinite.