Propositional Logic, II

Proof by Contradiction

Proof by Cases

Proof by Contradiction

Theorem: \( \sqrt{2} \) is irrational.
Proof (by contradiction):

• Suppose \( \sqrt{2} \) was rational.
• Choose \( m, n \) integers without common prime factors (always possible) such that 
  \[
  \sqrt{2} = \frac{m}{n}
  \]
• Show that \( m \) & \( n \) are both even, a contradiction!

Proof by Contradiction

Theorem: \( \sqrt{2} \) is irrational.
Proof (by contradiction):

\[
\sqrt{2} = \frac{m}{n} \\
\sqrt{2} n = m \\
2n^2 = m^2
\]
so can assume \( m = 2l \)
\[
m^2 = 4l^2 \\
2n^2 = 4l^2 \\
n^2 = 2l^2
\]
so \( n \) is even.

Quickie

Proof assumes that
If \( m^2 \) is even, then \( m \) is even.
Why!

Team Problem

Problem 1
Proof by Truth Tables

DeMorgan’s Law

\( \neg (P \lor Q) \) is equivalent to \( \neg P \land \neg Q \)

<table>
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<tr>
<th>P</th>
<th>Q</th>
<th>( \neg (P \lor Q) )</th>
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Proof by Deductions

A student is trying to prove that propositions \( P \), \( Q \), and \( R \) are all true. She proceeds as follows. First, she proves three facts:

- \( P \) implies \( Q \)
- \( Q \) implies \( R \)
- \( R \) implies \( P \).

Then she concludes,

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Thus \( P \), \( Q \), and \( R \) are obviously all true.
```

Proposed Deduction Rule

From: \( P \) implies \( Q \), \( Q \) implies \( R \), \( R \) implies \( P \)

Conclude: \( P \), \( Q \), and \( R \) are true.

\[
\begin{align*}
(P \rightarrow Q),\ (Q \rightarrow R),\ (R \rightarrow P) \\
\therefore\ P \land Q \land R
\end{align*}
\]

Sound Rule?

Conclusion true whenever all antecedents true.

\[
\begin{align*}
P \rightarrow Q & \quad Q \rightarrow R & \quad R \rightarrow P \\
\therefore\ P \land Q \land R
\end{align*}
\]
Sound Rule?
Conclusion true whenever all antecedents true.

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Reasoning by Cases
Quicker proof of unsoundness than from truth tables

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Quicker by Cases

\[
P \rightarrow Q, \quad Q \rightarrow R, \quad R \rightarrow P
\]

\[
P \land Q \land R
\]

Case 1: \( P \) is true. Now, if antecedents are true, then \( Q \) must be true (because \( P \) implies \( Q \)). Then \( R \) must be true (because \( Q \) implies \( R \)). So the conclusion \( P \land Q \land R \) is true. This case is OK.
**Quicker by Cases**

\[
P \rightarrow Q, \quad Q \rightarrow R, \quad R \rightarrow P
\]

\[
P \wedge Q \wedge R
\]

**Case 2:** \(P\) is false. To make antecedents true, \(R\) must be false (because \(R\) implies \(P\)), so \(Q\) must be false (because \(Q\) implies \(R\)). This assignment does make the antecedents true, but the conclusion \(P \wedge Q \wedge R\) is (very) False. This case is not OK.

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**Goldbach Conjecture**

Every even integer greater than 2 is the sum of two primes.

Evidence:

\[
\begin{align*}
4 &= 2 + 2 \\
6 &= 3 + 3 \\
8 &= 5 + 3 \\
\vdots \\
20 &= ? \quad 13 + 7
\end{align*}
\]

---

**Goldbach Conjecture**

True for all even numbers with up to 13 digits! (Rosen, p.182)

It remains an OPEN problem: no counterexample, no proof. UNTIL NOW!…

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**Goldbach Conjecture**

The answer is on my desk! (Proof by Cases)

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**Team Problem**

Problems 2 & 3