Predicate Logic

Quantifiers $\forall$, $\exists$

Predicates

$P(x, y) ::= [x + 2 = y]$

$x = 1$ and $y = 3$: $P(1,3)$ is true

$x = 1$ and $y = 4$: $P(1,4)$ is false

$\neg P(1,4)$ is true

Quantifiers

$\forall x$ For ALL $x$

$\exists y$ There EXISTS some $y$

Team Problems

Problems 1 & 2
Poet: "All that glitters is not gold."

$\forall x \ G(x) \rightarrow \neg Au(x)$

No!: gold glitters like gold

Math vs. English

Poet: "There is season for every purpose under heaven"

$\exists s \in \text{season} \quad \forall p \in \text{purpose} \quad s \text{ is the season for } p$

No!

Math vs. English

Poet: "There is season for every purpose under heaven"

$\forall p \in \text{purpose} \quad \exists s \in \text{season} \quad s \text{ is the season for } p$

(Poetic license again.)

Math vs. English

Poet: "All that glitters is not gold."

$G \quad \text{necessarily}$

$Au$

$\neg \left[ \forall x \ G(x) \rightarrow \neg Au(x) \right]$

(Poetic license)

Math vs. English

Poet: "There is season for every purpose under heaven"

$\forall p \in \text{purpose} \quad \exists s \in \text{season} \quad s \text{ is the season for } p$

(Poetic license again.)

Propositional Validity

$(A \rightarrow B) \lor (B \rightarrow A)$

True no matter what the truth values of $A$ and $B$ are

Predicate Calculus Validity

$\forall z \ [Q(z) \land P(z)]$

$\rightarrow \ [\forall x \ Q(x) \land \forall y \ P(y)]$

True no matter what
• the Domain is,
• the predicates are.
Not Valid
\[\forall z \ [Q(z) \lor P(z)] \rightarrow [\forall xQ(x) \lor \forall yP(y)]\]

Proof: Give countermodel, where \[\forall z \ [Q(z) \lor P(z)]\] is true, but \[\forall xQ(x) \lor \forall yP(y)\] is false.
Namely, let domain ::= \{e, \pi\},
\[Q(z) ::= [z = e]\],
\[P(z) ::= [z = \pi]\].

Validities
\[\forall z \ [Q(z) \land P(z)] \rightarrow [\forall xQ(x) \land \forall yP(y)]\]

Proof strategy: We assume
\[\forall z \ [Q(z) \land P(z)]\]
to prove
\[\forall xQ(x) \land \forall yP(y)\].

Predicate Inference Rule
\[
\frac{Q \rightarrow P(c)}{Q \rightarrow \forall x.P(x)}
\]
(providing \(c\) does not occur in \(Q\))
Universal Generalization (UG)

Validities
\[\forall z \ [Q(z) \land P(z)] \rightarrow [\forall xQ(x) \land \forall yP(y)]\]

Proof: Assume \[\forall z \ [Q(z) \land P(z)]\].
So \([Q(z) \land P(z)]\) holds for all \(z\) in the domain.
Now let \(c\) be some domain element. So
\([Q(c) \land P(c)]\) holds, and therefore \([Q(c)]\) by itself holds.
But \(c\) could have been any element of the domain.
So we conclude \[\forall xQ(x)\].
(by UG)
We conclude \[\forall yP(y)\] similarly. Therefore,
\[\forall xQ(x) \land \forall yP(y)\] QED.

More Validities
\[\forall x \ [P(x) \lor A] \leftrightarrow [\forall x P(x)] \lor A\]
(providing \(x\) not in \(A\))
\[\neg \forall x P(x) \leftrightarrow [\exists x \neg P(x)]\]

Team Problems
Problems
3 & 4