Graphs:
Handshaking, Connectivity & Trees

Possible Graph?
Is there a graph with vertex degrees 2, 2, 1?

NO!

An Impossible Graph

2 + 2 + 1 = odd

The Handshaking Lemma:
The sum of degrees must be even.

\[ 2 | E | = \sum_{v \in V} \deg(v) \]

The Handshaking Lemma

Proof: Each edge contributes 2 to the sum on the right

Connectedness

Vertices \( v, w \) are connected iff there is a path starting at \( v \) and ending at \( w \).

A graph is connected iff every two of its vertices are connected
**Paths & Simple Paths**

*Lemma:* The shortest path connecting two vertices is simple!

*Simple* path: all $v_i$ different

**Cycles & Simple Cycles**

Cycle: $v_0, v_1, \ldots, v_n, v_0$ for $n \geq 1$

where $v_i = -v_{i+1}$ is an edge for all $i$ s.t. $0 \leq i < n$

and $v_n = -v_0$

*Simple* cycle: all $v_i$ different and $n \geq 2$

*same cycle:* $v_2, v_3, v_0, v_1, v_n, v_{n-1}, \ldots, v_3, v_2$

**Connected Components**

The more connected components, the more broken up the graph is.

The connected component of vertex $v$ is

$\{w \mid v$ and $w$ are connected$\}$

Same as the block $[v]$ of the connectedness relation

**Cut Edges**

An edge is a *cut edge* if removing it from the graph disconnects two vertices.
Cut Edges and Cycles

**Lemma:** An edge is a cut edge iff it is not part of a simple cycle.

**Proof:** Class problem.

Cut Edges

Fault-tolerant design:

- In a tree, every edge is a cut edge (bad)
- In a mesh, no edge is a cut edge (good)

Tradeoff edges for failure tolerance

$k$-Connectedness

**Def:** $k$-connected iff remains connected when any $k-1$ edges are deleted.

Example:

$K_n$ is $(n-1)$-connected

Trees

A *tree* is a connected graph with no cycles.

More Trees
Other Tree Definitions

Lemma: A tree is a connected graph with $n$ vertices and $n - 1$ edges.

Lemma: A tree is an edge-minimal connected graph on a set of vertices.

Lemma: A tree is a graph with a unique path between any 2 vertices.

Be careful with these definitions

A tree is a graph with $n$ vertices and $n - 1$ edges??

NO:

Spanning Tree

Def: A subgraph that is a tree with all the vertices.

Always exists: any minimum-edge-size, connected subgraph on all the vertices

Problems

4 & 5