In-Class Problems Week 10, Mon.

Problem 1.
Recall that for functions $f, g$ on $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|.$$  \hfill (1)

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is $O()$ of the other, indicate the smallest nonnegative integer $c$, and for that smallest $c$, the smallest corresponding nonnegative integer $n_0$ ensuring that condition (1) applies.

(a) $f(n) = n^2$, $g(n) = 3n$.
(b) $f(n) = (3n - 7)/(n + 4)$, $g(n) = 4$
(c) $f(n) = 1 + (n \sin(n\pi/2))^2$, $g(n) = 3n$

Problem 2.

(a) Indicate which of the following asymptotic relations below on the set of nonnegative real-valued functions are equivalence relations (E), strict partial orders (S), weak partial orders (W), or none of the above (N).

- $f \sim g$, the “asymptotically equal” relation.
- $f = o(g)$, the “little Oh” relation.
- $f = O(g)$, the “big Oh” relation.
- $f = \Theta(g)$, the “Theta” relation.
- $f = O(g)$ AND NOT($g = O(f)$).

(b) Indicate the implications among the assertions in part (a). For example,

$$f = o(g) \text{ implies } f = O(g).$$

Problem 3.

False Claim.

$$2^n = O(1).$$ \hfill (2)

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.
Bogus proof. The proof is by induction on $n$ where the induction hypothesis, $P(n)$, is the assertion (2).

Base case: $P(0)$ holds trivially.

Inductive step: We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n + 1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all $n$. That is, the exponential function is bounded by a constant.

Supplemental problems

Problem 4.
Assign true or false for each statement and prove it.

- $n^2 \sim n^2 + n$
- $3^n = O(2^n)$
- $n^{\sin(n\pi/2) + 1} = o(n^2)$
- $n = \Theta\left(\frac{3n^3}{(n+1)(n-1)}\right)$

Problem 5.
Give an elementary proof (without appealing to Stirling’s formula) that $\log(n!) = \Theta(n \log n)$. 